GDAŃSK UNIVERSITY OF TECHNOLOGY
FACULTY OF MANAGEMENT AND ECONOMICS

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MASTER'S THESIS

Title of thesis: Expected Utility Hypothesis – its origin and development

Title of thesis (in Polish): Hipoteza Oczekiwanej Użyteczności – jej pochodzenie oraz rozwój

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Abstract

Niniejsza praca przedstawia powstanie Hipotezy Oczekiwanej Użyteczności wraz z jej tłem historycznym oraz opisuje jej wczesny rozwój. Pierwszy rozdział prezentuje historię oraz rozwiązania Paradoksu Petersburskiego, który jest blisko związany z Teorią Oczekiwanej Użyteczności von Neumanna-Morgensterna stanowiącą główne zagadnienie pracy. W drugim rozdziale, Teoria Oczekiwanej Użyteczności jest dogłębnie przeanalizowana w oparciu o oryginalną publikację von Neumanna-Morgensterna (Von Neumann and Morgenstern, 1953). Ich aksjomatyzacja wraz z interpretacją poszczególnych aksjomatów kończy pierwszą część tego rozdziału. W następnej części, zaprezentowane zostały podstawowe zagadnienia z zakresu teorii decyzji, które wykorzystują pojęcie oczekiwanej użyteczności. W ostatniej części, rozdział trzeci poświęcony jest krytyce dotyczącej teorii oraz analizie paradoksu Allais'a wraz z opisem współczesnych psychologicznych przyczyn łamania założeń Teorii Oczekiwanej Użyteczności.

Słowa kluczowe: Hipoteza Oczekiwanej Użyteczności; Teoria Oczekiwanej Użyteczności; Paradoks petersburski; Zakład Pascala; Super-Paradoks petersburski; Twierdzenie o użyteczności oczekiwanej Von Neumanna-Morgensterna; Paradoks Allais; Krzywa obojętności

Dziedzina nauki i technologii, według wymagań **OECD**: 5.2 Ekonomia; 5.2 Biznes i Zarządzanie; 1.1 Matematyka stosowana

Abstract

The following thesis introduces the origin of Expected Utility Hypothesis with its historical background and describes its early development. In the first chapter, the history and solutions to St. Petersburg Paradox are presented, as it is closely related to the von Neumann-Morgenstern Expected Utility Theory which is the core notion of the thesis. In the second chapter, Expected Utility Theory is thoroughly analysed basing on the original publication of von Neumann and Morgenstern (Von Neumann and Morgenstern, 1953). Their axiomatisation together with the interpretation of the axioms concludes the first part of the chapter. Next, the basic notions of decision theory utilising the concept of expected utility is presented. Finally, the third chapter is devoted to the presentation of Maurice Allais's critique of the theory and the analysis of his paradox with the brief description of modern psychological reason for Expected Utility Theory violations.

Keywords: Expected Utility Hypothesis; Expected Utility Theory; St. Petersburg Paradox; Pascal's Wager; Super-Petersburg Paradox; Von Neumann-Morgenstern utility theorem; Allais Paradox; Indifference curves

Field of Science and Technology, as required by **OECD**: 5.2 Economics; 5.2 Business and Management; 1.1 Applied mathematics

Introduction

The subject of Expected Utility Hypothesis is very interesting due to its many different dimensions, its interdisciplinary character and the fact that it is still a topical subject in the field of modern economy. The beginning of the Expected Utility Hypothesis dates back to the eighteenth century and since then it has evolved from a mainly mathematical and to some extent philosophical notion to an important concept researched by modern economists and psychologists (Dutka, 1988) . Many different theories have arisen, taken advantage of, or developed thanks to it. To mention a few of them: mathematical branches such as probability theory or game theory; more connected with economics such as decision theory and, last but not least, behavioural economics being an interdisciplinary study of both economics and psychology. The Expected Utility Hypothesis can be considered as a metaphorical bridge between mathematics and economics. Its creation was an offspring of real life problems and considerations and it has been raising questions since then. The best example of topicality of the subject is Prospect Theory, created in 1979 (Kahneman and Tversky, 1979), and developed in 1992 by Daniel Kahneman and Amos Tversky and later Cumulative Prospect Theory (CPT) developed by Kahneman, for which he was given a Noble prize in economics in 2002.

The aim of the following thesis is to present in detail the historical background and the origin of the Expected Utility Hypothesis starting from the St. Petersburg Paradox and Pascal's Wager, through the creation of axiomatic basis for Expected Utility Theory developed by John von Neumann and Oskar Morgenstern, followed by the brief analysis of Maurice Allais's critique of von Neumann's and Morgenstern's achievements (Allais, 1953), ending with thorough presentation of Allais Paradox and its contemporary psychological implications (Iqbal, 2013). The work is focused on the early sources and key publications concerning the subject (Von Neumann and Morgenstern, 1953) (Allais, 1953) (Tversky, 1975) (MacCrimmon and Larsson, 1979). Through the historical background and detailed step-by-step analysis of the basics of the aforementioned notions, it aims to give a solid basis for further research and understanding of contemporary theories which are based or related in any way to the Expected Utility Hypothesis.

The thesis is divided into three main chapters. In the first chapter, the historical background, the origin and the circumstances of publication of the St. Petersburg Paradox are described. The chapter ends with a presentation of solutions to the paradox and its relation to the creation of Expected Utility Hypothesis. The second chapter is focused on the creation process of axioms and an early beginning of Expected Utility Theory with the detailed analysis of the most important publication regarding the theory i.e. *Theory of Games and Economic Behavior* by von Neumann and Morgenstern (Von Neumann and Morgenstern, 1953). Furthermore, in the second part of the second chapter, the interpretation of the axioms is followed by the analysis of decision theory with the emphasis put on decision making under certainty and uncertainty with the application of Expected Utility Theory notions. In the last chapter, the critique of Expected Utility Theory by one of its fiercest opponents Maurice Allais is presented. This part is based on his famous article *"Le Comportement de l'Homme Rationnel devant le Risque: Critique des Postulats et Axiomes de l'Ecole Americaine"* (Allais, 1953). It is followed by the description of the paradox itself with the help of indifference curve analysis (Machina, 1987). The thesis is concluded with the description of the most common psychological causes of Expected Utility Hypothesis violations.

Although the subject of the thesis has not been covered during the master degree studies, it is closely related to many subjects and problems connected with economics and management. Despite the fact that some notions such as marginal utility of money or the method of indifference curves analysis were introduced, they can merely be considered as an elementary introduction to some more advanced problems described in this work. Therefore, the process of writing the thesis was impeded. It required additional preparation and self-learning in order to get to know particular notions considered as elementary in the researched field. Due to the aforementioned reasons, the theoretical character of the work was chosen intentionally. The thesis presents notions on which the literature is either limited or unavailable in Poland (recent publication which is easily available (Dobrowolski, 2014)). Hence, the work is aimed to be a comprehensive introduction of these notions, which is available in singular piece of writing. Since the scope of the work is wide, any additional empirical research would exceed the size of a master thesis.

1 St. Petersburg Paradox

1.1 Introduction

Over three hundred years ago, on 9th September 1713, a paradox which significantly changed the view on money utility was created. What might have been called an ordinary puzzle at that time is nowadays referred to as the St. Petersburg Paradox and is still discussed and analysed in many scientific publications and literature (Weber, 1998) (Joyce, 2011) (Seidl, 2013). Three years ago, on the 300th anniversary of the paradox, Christian Seidl published an article "The St. Petersburg Paradox at 300", which is a great contemporary overview of the problem with a solid mathematical analysis (Seidl, 2013).

While most of the historical figures connected with the birth of the paradox were mathematicians, the paradox had much wider impact on the world's perception and mathematics itself. Not only did it create discussion in the branches of mathematics such as probability theory, statistics and later game theory, but also caught the attention of philosophers and more recently economists all around the world. However, perhaps most importantly, it gave rise to the expected utility theory and changed the view on mathematical expectation in relation to the real world.

In the following chapter, the history of the St. Patersburg Paradox will be presented together with the details regarding its publication. Furthermore, the most common solutions with examples will be discussed prior to the description of crucial input of the paradox to the creation of Von Neumann-Morgenstern utility theory.

1.2 Historical background and publication of the St. Petersburg Paradox

1.2.1 Mathematical views and believes prior to the paradox

Probability theory which is perhaps considered the "most applicable" branch of mathematics to the real world by laymen, arose in the seventeenth century due to the popularity of gambling and games of chance. At that time in France, among aristocrats one of the most popular ways of spending their free time was gambling. One of them had a great impact on development of probability theory due to his immense pragmatism. His name was Antoine Gombaud, also known as Chavalier de Mere, a French writer, who asked two of the most famous mathematicians of his time, Pascal and Pierre de Fermat, for a mathematical guidance in gambling (Dutka, 1988). This request resulted in later correspondence between the two mathematicians regarding gaming problems. What is particularly interesting, the term "probability" was never explicitly used yet.

The main suggestion Pascal and Fermat made to Gombaud was to use the expected value of the winnings. This was a very important remark since it gave rise to a viewpoint on making rational decisions when faced with a risk (events probable to unknown extent) by the use of mathematical expectation i.e. expected value. The Pascal-Fermat correspondence focused mainly on two problems presented by Gombaud which were already known to the gamblers of that time. The former concerned the problem of fair division of stakes in case of several players taking part in an interrupted series of games of chance. The latter, concerned the problem of obtaining the given sum within the given number of throws of a die and its mathematical assessment of a player's advantage. Even though Pascal and Fermat did not use the term "probability", their descriptions of terms concerning anticipated profits or loses can be interpreted with the use of modern notions of mathematical expectation.

The Pascal-Fermat correspondence was followed by the work of Christiaan Huygens who learnt about it while visiting Paris in 1655. Regrettably, he did not have a chance to meet either of two great mathematicians. After he returned to Holland, he wrote an important tract on probability in Dutch, which was later translated into Latin. The Latin version was well received by the mathematicians of that time and as a result was translated into many other languages. Huygens in his work focused mainly on the concept of expected value, however, without formally defining it. Instead, he used descriptive terms such as "the worth of the chance" or "it is worth to me". Moreover, the equivalent of modern fair game can be found in his work, which is characterised by the same expectation of profit for each player taking part in it. Finally, what is worth mentioning, the fact is this tract remained the only widely available work on probability for almost half a century.

Naturally, the probability theory did not arise only due to the interest in gambling and games of chance. During the seventeenth century there was a substantial growth in many sectors involving large sums of money (payouts) and the probability of some related contingent events occurring. To list some of them, starting from the ones that are the most related to gaming problems, there are national lotteries, any kind of insurance policies (life, entrepreneurial, marine) or tontines[1]. The knowledge of conclusions from Pascal-Fermat correspondence

[1] TONTINE - was a system of life insurance named after an Italian banker Lorenzo Tonti born at Naples in the beginning of the 17th century. He settled in France about 1650. In 1653 he proposed to Cardinal Mazarin a new scheme for promoting a public loan. His suggestion was to subscribe a total of 1,025,000 livres in ten portions (102,500 livres each) by ten classes of subscribers. The division of the classes was as follows: the first class consisted of persons under 7, the second of persons above 7 and under 14, and so on to the tenth, which consisted of persons between 63 and 70. The annual fund of each class was to be divided among the survivors of that class.

and rising popularity of Christian Huygens tract on probability in the second part of the seventeenth century resulted in first attempts to incorporate the newly formed idea of quantitative probability to other areas such as demographics, annuities and abovementioned insurance services. The forthcoming successes in many areas connected with applying the notions of early probability theory into the real life problems resulted in further interest and development of that field.

1.2.2 Pascal's Wager and the first occurrence of "infinite gain" concept

Prior to the discussion about the origin of St. Petersburg Paradox itself, there is one more significant event which should be mentioned due to one important correlation. Aforementioned mathematician Blaise Pascal, after his second religious conversion in November 1654, mostly abandoned research in the field of mathematics and physics and focused more on theology and philosophy. Pascal was never a supporter of the idea of proving God's existence as he believed in superiority of believe over reason - "the heart has its reasons which reason does not know". However, he created a very interesting and provoking argument especially for non-believers in form of a wager. Nowadays refereed to as Pascal's Wager, it is one of the most famous arguments in philosophical theology.[2]

The wager is based on the Pascal's assumption that one either believes in God or not, without any other alternative. Hence, it can by regarded as a lottery with two possible "choices" and two possible "cases" with unknown probabilities. Let E be the case associated with God's existence and p the probability of the case E occurring (the wager assumes that p is positive - it might be infinitesimal but not equal to zero). Then, let nE denote the case of God's non-existence. The two possible choices are named as follows: B - to believe; and nB - not to believe. Having established the notation, a simple table summarizing the wager and potential "utilities" (in the sense of outcomes and gains associated with them) can be made, where u_1, u_2, u_3 and u_4 denote utilities for every one of four possible outcomes (Tabarrok, 2000).

	E	nE
B	∞	u_1
nB	u_2	u_3

Table 1: Pascal's Wager. Source: *Own compilation.*

Furthermore, on the death of the last individual the capital was to fall to the state.

[2]The aim of the following notation is to present the Pascal's Wager in the form of mathematical notation in accord with probability and game theory in order to highlight the aspect of wager's infinite gain.

Pascal argued that if one accepts God's existence and believes, then he might expect eternal salvation while not loosing anything in his life. Hence, the expected utility of such case is infinite. On the other hand, if one rejects God's existence and ultimately turns out to be wrong, than all the life's work might be lost. Regardless of the case, it might be assumed that each and every utility level u_1, u_2 and u_3, apart from the case of God's existence and actual believing, is finite. Having made this assumptions, it is easy to calculate the expected utilities of the two choices: B - believing in God; and nB - not believing.

$$E(B) = p \times \infty + (1-p) \times u_1 = \infty$$

$$E(nB) = p \times u_2 + (1-p) \times u_3 = u_4$$

Even though the numerical values of utility levels u_1, u_2, u_3 and u_4 are unknown and impossible to calculate, it is certain that they are finite. Hence:

$$u_4 \ll \infty \Leftrightarrow E(nB) \ll E(B)$$

According only to the calculated expected utility, every rational human being should believe in God.

Although the wager is first and foremost the problem considered by philosophers and theologians not mathematicians, it outlines the possibility of infinite gain in a game of chance, which is a main problem of the St. Petersburg Paradox.

1.2.3 Nicolas Bernoulli's five problems to Pierre Rémond de Montmort

The problem of St. Petrsburg Paradox was created by Nicolas Bernoulli (1687 - 1759), however, in a different form from the one known today. Nicolas was a nephew of the famous Jacob Bernoulli (1655 - 1705) - the creator of a treatise on probability theory, which is considered a milestone in this filed. The work was mainly written in 1690s, nevertheless, it was left unfinished for Jacob's death in 1705. Nicolas Bernoulli dedicated himself to publication of his uncle's unfinished manuscript, which he realised in 1713. Even though he became a professor of jurisprudence in Basel, he did not abandon the interest in mathematics.

The first version of the paradox can be found in a letter sent by Nicolas Bernoulli to French mathematician Pierre Rémond de Montmort (1678-1719). Bernoulli, who had a long correspondence with de Montmort, sent him a letter with five problems on 9th September 1713. Later, they were published in de Montmort's second edition of his famous book on games of hazard.

Only the last two problems (the fourth and the fifth) are important for St. Petersburg Paradox considerations.

The fourth problem

The fourth problem was described as follows:

Player A promises to give a crown to player B if: with an ordinary die he gets six points on the first throw, two crowns if he gets the six on the second throw, three crowns if he gets this point on the third throw, four crowns if he gets it on the fourth, and so on; B's expectation is required.

A solution to this problem is easily obtainable by the use of expected value $\mathbb{E}X = \sum_{i=1}^{n} p_i x_i$. The expected value for discrete cases (such as consecutive throws of a die) is defined as the sum of products consisting of values of a random variable (in our case payoffs) and associated probabilities. Let us remark that if the random variable X is infinite but countable, then n can be replaced with ∞ provided that such a sum converges absolutely.

The probability of getting a "6" on the first throw is straightforward and equals $\frac{1}{6}$. The probability of getting "6" on the second throw (given that "6" was not obtained during the first roll) is equal to $\frac{5}{6} \cdot \frac{1}{6}$. The probability of getting "6" for the first time on the third throw is equal to $\frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6}$. From the above examples we can clearly see and create the general formula for probability of tossing "6" for the first time on the n^{th} throw.

$$\underbrace{\frac{5}{6} \cdot \frac{5}{6} \cdot \ldots}_{n-1} \cdot \frac{1}{6} = \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6} = \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{n-1}$$

When it comes to the values (payoffs) associated with the probabilities, the formula is very simple. As player A is to give player B number of crowns equal to the number of the throw on which "6" is thrown for the first time, it is simply denoted by n. Hence, the expected value can be created. Due to the fact that theoretically "6" can be tossed for the first time as late as we can imagine, the sum in expected value goes to infinity, which makes it a series. Summarising above explanations, the expected value for the fourth problem is created in the following way[3]:

$$\sum_{n=1}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{n-1} \cdot n = \frac{1}{6}\sum_{n=1}^{\infty}\left(\frac{5}{6}\right)^{n-1} \cdot n = \frac{1}{6}\sum_{n=1}^{\infty} n\left(\frac{5}{6}\right)^{n-1} \tag{1.1}$$

Now, let us focus on calculating the sum of the series in the above equation 1.1. This step requires the use of the generalised formula of the geometric series and differentiation of power series:

$$\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (x^n)' = \left(\sum_{n=0}^{\infty} x^n\right)' \overset{\curvearrowright}{=} \left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$$

The first two transitions of the above equation are straightforward and only use the basic principles of differential calculus such as the sum rule in differentiation (taking advantage of linearity of differentiation) (Jakubowski and Sztencel, 2011). The third transition, indicated in the above equation by the symbol $\curvearrowright$, utilises the well-known sum of the geometric series. In order to explain how this sum was obtained, we need to recall the simplified definition of geometric series.

Definition 1 (Geometric series)
A geometric series $\sum_n a_n$ is a series which terms form a geometric progression. The ratio of two consecutive terms of such series a_{n+1}/a_n is constant. Hence, a geometric series can be represented using only two terms: common ratio[4] $x = a_{n+1}/a_n$ and the first term of the series denoted by a. When considering the simplest case of common ratio equal to a constant x, the terms a_n have the following form: $a_n = a_0 x^n$. Let $a_0 = 1$, than the geometric sequence $\{a_n\}_{n=0}^{m}$ with constant $|x| < 1$ is given by the sum:

$$S_m = \sum_{n=0}^{m} a_n = \sum_{n=0}^{m} x^n \tag{1.2}$$

Having defined the geometric series we can now resume our explanation of the transition indicated by $\curvearrowright$. Let us notice that:

$$S_m \equiv \sum_{n=0}^{m} x^n = 1 + x + x^2 + \ldots + x^m \tag{1.3}$$

We are about to perform several operation on the above equation in order to prove

[3]Please note that on the second transition in the equation below, the fraction $\frac{1}{6}$ can be taken away from the series (infinite sum) due to the fact that it is a constituent independent from n. The last transition is made purely for aesthetic reason to make it easier to see the solution of summing the series later on.

[4]Common ratio is usually denoted by the letter r in the literature, however, for the sake of consistency the letter x will be used.

the transition $\curvearrowright$. Let us multiply both sides of the equation 1.3 by x, which results in:

$$xS_m = x + x^2 + x^3 + \ldots + x^{m+1} \tag{1.4}$$

Now we are going to subtract equation 1.4 from equation 1.3, which gives:

$$(1-x)S_m = \big(1 + x + x^2 + \ldots + x^m\big) - \big(x + x^2 + x^3 + \ldots + x^{m+1}\big) = 1 - x^{m+1} \tag{1.5}$$

Hence, the sum S_m can be presented in the following way:

$$S_m \equiv \sum_{n=0}^{m} x^n = \frac{1 - x^{m+1}}{1 - x} \tag{1.6}$$

Finally, now it is sufficient to notice that for $-1 < x < 1$ and $m \to \infty$ the sum converges and gives:

$$S \equiv S_\infty = \sum_{n=0}^{\infty} x^n = \frac{1}{1 - x} \tag{1.7}$$

□

Ultimately, it suffices to use the result of the above proof 1.7 and previous considerations to calculate the expected value for the fourth problem:

$$\frac{1}{6}\sum_{n=1}^{\infty} n \left(\frac{5}{6}\right)^{n-1} = \frac{1}{6}\frac{1}{\left(1 - \frac{5}{6}\right)^2} = 6$$

The fifth problem

The fifth and the last problem was described as follows:

The same is required if player A promises player B to give the crowns in the progression:

a) 1, 2, 4, 8, 16, ... or

b) 1, 3, 9, 27, ... or

c) 1, 4, 9, 16, 25, ... or

d) 1, 8, 27, 64, ...

instead of 1, 2, 3, 4, 5, ... as before.

The above progressions can be expressed analogically to the fourth problem i.e.

a) 1, 2, 4, 8, 16, ... $\longmapsto$ $a_n = 2^{n-1}$ $\Longrightarrow$ $\frac{1}{6}\sum_{n=1}^{\infty} 2^{n-1}(\frac{5}{6})^{n-1}$

b) 1, 3, 9, 27, ... $\longmapsto$ $a_n = 3^{n-1}$ $\Longrightarrow$ $\frac{1}{6}\sum_{n=1}^{\infty} 3^{n-1}(\frac{5}{6})^{n-1}$

c) 1, 4, 9, 16, 25, ... $\longmapsto$ $a_n = n^2$ $\Longrightarrow$ $\frac{1}{6}\sum_{n=1}^{\infty} n^2(\frac{5}{6})^{n-1}$

d) 1, 8, 27, 64, ... $\longmapsto$ $a_n = n^3$ $\Longrightarrow$ $\frac{1}{6}\sum_{n=1}^{\infty} n^3(\frac{5}{6})^{n-1}$

Solving the fifth problem is not that easy due to the fact that expected value for the first two cases (a & b) does not exist. It can be explained by noticing that the first two series i.e. $\frac{1}{6}\sum_{n=1}^{\infty} 2^{n-1}(\frac{5}{6})^{n-1}$ and $\frac{1}{6}\sum_{n=1}^{\infty} 3^{n-1}(\frac{5}{6})^{n-1}$ are divergent. On the other hand, for the last two series (c & d) i.e. $\frac{1}{6}\sum_{n=1}^{\infty} n^2(\frac{5}{6})^{n-1}$ and $\frac{1}{6}\sum_{n=1}^{\infty} n^3(\frac{5}{6})^{n-1}$, expected value can be obtained as these two series are convergent.[5]

P. R. de Montmort did not find Bernoulli's problems interesting and in reply to the author he suggested that these problems can be easily solved by the method of summation of the series[6] developed by aforementioned Jacob Bernoulli - Nicolas's deceased uncle (Seidl, 2013). On 20th February 1714 Bernoulli sent another letter with his solutions of the problems. For the fourth problem he correctly summed the convergent series achieving the solution equal to 6. However, when he tried to apply the method to the first case of the fifth problem he achieved the result equal to $-\frac{1}{4}$ in effect summing the divergent series. He considered it as a contradiction, which resulted in some fallacious attempts to solve it. The contemporary viewpoint on summation of the series stands that a series $\sum a_N$ is said to be convergent when the sequence S_N of partial sums has a finite limit. Otherwise, if the limit of S_N is infinite or does not exist, the series is said to be divergent. When the limit of partial sums exists, it is called the sum of the series ($\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=0}^{N} a_n$). Hence, the divergent series cannot be summed as Bernoulli did. It is crucial to point out that the contemporary view on the sum of infinite series (sums of infinite series regardless of its convergence are treated as mathematical conventions[7]) was not developed until the second part of the nineteenth century.

Despite the incorrect tries of solving the contradiction, the conclusion made by Nicolas is important for further development of the problem. He argues that the fair value of expectation does not have to be the sum of constituent expectations as some events with very small probability should be disregarded and treated as null. Nevertheless, it is important to realise that however insignificant the probability may seem, the amount associated with it might alter the final result significantly. This is a very important concept which will be

[5]The proofs of both divergence and convergence of the series will not be discussed in this work as they do not contribute significantly to the main problem of the thesis.

[6]The method will not be described here as it is not crucial for understanding the main problem of the chapter. For further reading on the subject please refer to: *Bernoulli Summation Formulas*, *Bernoulli Numbers*, *Euler-Maclaurin Summation Formula*.

[7]A mathematical convention is a fact, name, notation, or usage which is generally agreed upon by mathematicians. An example of mathematical convention can be a factorial of zero i.e. $0! = 1$.

discussed in greater detail in the following chapters. Bernoulli and many of his successors regarded the paradox as a discrepancy between widely accepted use of expected value for valuation of games of chance and the actual anticipation of the return in the game. In the final reply to Nicolas, de Montmort accepted his argument but tended to support the validity of expected value. However, he suggested in diplomatic way that the only qualified person for further research in this matter was Bernoulli himself. Despite Nicolas's trials to keep de Montmort engaged with the development of the problem, the latter did not contribute significantly before he passed away in 1719.

A very important and meaningful remark on the matter was made by J. Dutka (1988) who concluded that the fourth and the fifth problem developed by Nicolas Bernoulli led to a comparison of formal mathematical results and the actual human behaviour in described situations. *"The significance of the results cannot simply be judged on the basis of whether they are correct deductions from certain initial mathematical assumptions. If the results are to be compared with the actual behaviour of people, the initial mathematical assumptions must also be examined to determine whether they are meaningful and valid in the real world."* (Dutka, 1988). He also pointed out that with regard to the fourth problem two aspects should be considered: the ability of Player A to pay any sum of money which Player B wins, and the possibility of unlimited number of throws of a die. Furthermore, the fifth problem complicates the considerations even more as it introduces the concept of solution involving "infinite mathematical expectation" which was seen by Nicolas Bernoulli and most probabilist of the eighteenth century as legitimate mathematical concept. All of the above resulted in a deeper consideration of the term "infinity" itself.

In the seventeenth and eighteenth centuries the term "infinity" was associated with a number, however, larger than any finite number one could think of. This had been a prevailing notion until it was replaced in the nineteenth century when the idea of "infinity" was introduced as a kind of limiting process in mathematical analysis. The former view was contested by i.a. Carl Friedrich Gauss (1777-1855) and Augustin Louis Cauchy (1789-1857). In a frequently quoted letter to the astronomer Heinrich Christian Schumacher (1780-1850) on 12th July 1831, Gauss contested the prevailing view on the term "infinity":

"... I protest ... against the use of an infinite quantity as an actual entity, which is never allowed in mathematics. The infinite is only a façon de parler[8] *in which one really speaks of limits to which certain ratios come as near as desired, while others are allowed to increase unrestrictedly."*

Having established the time-line and views on important mathematical

[8]From French "façon de parler" means - "way of speaking", "manner of speech"

notions, the analysis of the Nicolas Bernoulli's fifth problem and its transition to St. Petersburg Paradox can be resumed. The problem presented to de Montmort can be presented from more general but at the same time formal viewpoint. Let the required expectation be expressed by the use of divergent infinite series of finite expectations:

$$\sum_{n=1}^{\infty} p(n)a(n) \tag{1.8}$$

The formula above is constructed in such a way that: $p(n)$ represents the probability of winning on the n^{th} trial; $a(n)$ represents the amount won; where $n = 1, 2, 3, \ldots, \{a(n)\}$ is an increasing sequence, while $\{p(n)\}$ is a decreasing sequence.

In reference to this formula, Nicolas Bernoulli suggested a substitution of sequence $\{p(n)\}$ with another sequence $\{\bar{p}(n)\}$ such that the newly created series:

$$\sum_{n=1}^{\infty} \bar{p}(n)a(n)$$

converges. The idea behind the sequence $\{\bar{p}(n)\}$ is to replace very small probabilities with zero. In fact it means "cutting the tail" of the sequence $\{p(n)\}$ for n exceeding some value m.

1.2.4 Gabriel Cramer's contribution to the problem

The form of the paradox which is known nowadays was developed by the Swiss mathematician Gabriel Cramer, who had the biggest impact on the development of the problem in its early stage. On the letter to Nicolas Bernoulli from the 21st of May 1728, he suggested an alternative solution to the problem described by the formula 1.8. In contrary to what was proposed by Nicolas, Cramer suggested an equivalent solution to substitute the sequence $\{a(n)\}$ with another sequence $\{\bar{a}(n)\}$ such that the series

$$\sum_{n=1}^{\infty} p(n)\bar{a}(n) \tag{1.9}$$

converges. However, one of the most crucial insights in the letter was the simplification of the Nicolas's fifth problem. Cramer suggested replacing the six-sided die with a two-sided (fair) coin and exchange the roles of Player A and Player B. As a result, if player A tosses the first head on the n^{th} trial after having tossed $n-1$ consecutive tails, he is given 2^{n-1} crowns by the player B, where $n = 1, 2, 3, \ldots$. It can be noticed that so far Player A's expectation can be formally expressed by the series 1.8 on page 16.

Having presented Cramer's simplification, the contemporary version of St. Petersburg Paradox later published by Daniel Bernoulli (described in detail in the following chapter) can be quoted:

Peter tosses a coin and continues to do so until it should land "heads" when it comes to the ground. He agrees to give Paul one ducat if he gets "heads" on the very first throw, two ducats if he gets it on the second, four if on the third, eight if on the fourth, and so on, so that with each additional throw the number of ducats he must pay is doubled. Suppose we seek to determine the value of Paul's expectation.

What Cramer considered a paradox was that, basing on calculations, Player A should pay Player B an infinite sum of money to play the game. As he argued, that appears to be an absurd since no reasonable person would pay 20 crowns to enter the game. His further reasoning has been quoted many times in the literature considering this subject:
"What is the reason for this difference between the mathematical calculation and the ordinary valuation? This is because mathematicians value money in proportion to the amount, wheras reasonable people value it in proportion to the use they can make of it."
The very similar reasoning was expressed by Daniel Bernoulli in reference to the version of the paradox quoted above.
"...Although the standard calculation shows that the value of Paul's expectation is infinitely great, it has ... to be admitted that any fairly reasonable man would sell his chance, with great pleasures, for twenty ducats."
Another remark, however a bit harsh, was made by a friend and correspondent of Cramer - a French naturalist G. L. L. Buffon (1707-1788).
"The miser is like the mathematician - both value money by its numerical quantity."
All this reasoning laid the ground for the later creation of expected utility hypothesis.

Cramer continued his deliberations on the problem by pointing out that what makes the mathematical expectation infinite is the possibility of winning enormous amount of money if the player does not toss a "head" until very late trial e.g. 100^{th} or 1000^{th} toss. Furthermore, he argued that for a sensible man neither should it be worth more nor yield more pleasure than if the amount to be won was limited by 10 or 20 million crowns. Later, basing on this assumption, he calculated the expectation in line with the series 1.9 on page 16 for the limited amount of $2^{24} = 16,777,216$ crowns. It is assumed that either Player A accepts that the maximum amount to be won is equal to 16,777,216 crowns (the payoff in case of tossing the first head on the 25^{th} trial) or that the capital of Player B is limited to this amount. Therefore, the sum of the series 1.9 (which is now limited) is obviously finite and equals 13 (it is a pure numerical value without a unit). Cramer called his result a "moral value of wealth", which by modern economists would be called utility of money. He had a very original view on that matter and was one of the first to formalise this concept.

Gabriel Cramer considered the achieved result as too big. Hence, he made an attempt to further decrease the "moral value of wealth" by suggesting an alternative assumption. He made a remark that even though 100 million yields more pleasure than 10 million, it is certainly not ten times as much. That is why, he suggested that moral value of wealth should be a square root of mathematical quantity. However, he stated that this is not the equivalent of the actual stake of the game since it should not be equal to moral expectation but rather equal to the regret for the loss of expected pleasure. This remark is of a great importance for economists as it reflects the notions of "principle of maximum regret", which was suggested in decision and economic theory more than two centuries later.

Nicolas Bernoulli did not agree with Gabriel Cramer's arguments. He still considered the utility from infinite sum as greater than utility gained from finite, however, very large sum. Moreover, he stated that arguments presented by Cramer did not explain why mathematical expectation was different from the ordinary estimate. He supported his believe by stating that a pragmatic view does not take into consideration the magnitude of potential winnings. Hence, a very small probability of wining a large sum is regarded as impossible but, on the other hand, the very large probability of wining a small sum is considered as almost certain. In Nicolas's view these two probabilities do not counterbalance each other in ordinary estimation. Gabriel Cramer did not accept Bernoulli's arguments either and it appears there was not further correspondence on that matter between the two of them.

What is particularly interesting from the modern viewpoint is the fact that judging from Nicolas Bernoulli's reasoning he would have never taken part in any kind of lottery which involved only small probabilities of winning huge amounts of money. Funnily enough, this was a view considered as legitimate more than three hundred years ago, while nowadays a considerable part of the society takes part in such lotteries on every day basis. Personally, I consider this fact as a proof that problems of decision theory and evaluation of risk and utility are still up to date and need to be researched further.

1.2.5 The origin of the name of the paradox and its first publication

After receiving Cramer's simplified version of his problem, Nicolas Bernoulli decided to familiarise with it his cousin Daniel Bernoulli, who was at that time a professor of mathematics at the University of St. Petersburg. On 27th October 1728 Nicolas sent to his cousin the fourth and the fifth problem (in Cramer's simplified version). At first Daniel Bernoulli was not interested in them and regarded them as very easy, though a bit paradoxical. In his reply he stated that there is a little probability that the game would last longer than 20 or 30 throws. Nicolas rejected his cousin's argumentation, which made Daniel reconsider the problem. Later, Daniel Bernoulli sent to Nicolas a memoir, which shed a new light on the problem. Daniel suggested that the initial fortune of a player should also be considered in order to determine his expectation. In regard to Daniel's suggestion, Nicolas believed that his ideas combined with Cramer's and Daniel's insights might lead to a more accurate way of disregarding small probabilities.

The name of the paradox is strictly connected with its publication and the circumstances. There seem to be no publications dated before 1738 on the problem apart from the statement in de Montmort's book about his correspondence with Nicolas Bernoulii in 1713. In 1731 Daniel Bernoulli submitted his memoir for publication in the *Commentarii* to the St. Petersburg Academy, where it was officially published seven years later in 1738. Thus, the St. Petersburg Paradox has derived its name from the place of its first ever publication.

Daniel in his memoir introduces very important hypothesis which is the basis of theory of marginal utility widely used in modern economy. He suggests that in order to determine the value of the risk for a particular individual the mathematical expectation of contingent events is not sufficient. Moreover, he implies that in reality the possibility to win a given amount of money is not equally important to different people, but is rather relative to their current wealth. He continues with formulating the following hypothesis:
"Now it is highly probable that any increase in wealth, no matter how insignificant, will always result in an increase in utility which is inversly proportional to the quantity of goods already possessed."
The above hypothesis can be denoted in mathematical terms as a following derivative:

$$dy = k\frac{dx}{x}$$

where dy indicates an increase in utility for an individual; x denotes his present wealth and dx receiving additional amount of money; and k which is the proportional factor (we demand $k > 0$) subjective for every individual. In order to find y the above formula should be integrated as follows:

$$y = k\int_a^x \frac{dx}{x} = k\ln\frac{x}{a} \tag{1.10}$$

where a (we demand $a > 0$) denotes the initial fortune.
Now let a be an initial fortune of a person who plays the game in which the amount a_n can be won with probability p_n for $n = 1, 2, 3, \ldots$ and $\sum_{n=1}^{\infty} p_n = 1$. His mathematical expectation is straightforward and equal to: $\sum_{n=1}^{\infty} p_n a_n$. However, according to Bernoulli's hypothesis, as he called it "mean utility", it is equal to:

$$k\sum_{n=1}^{\infty} p_n \ln\left[\frac{(a + a_n)}{a}\right]$$

assuming the series converges.
It is important to mention that Bernoulli's mean utility was later called "moral expectation" by Pierre Simon de Laplace (1749-1827).

Daniel's theory of moral expectation was rejected by Nicolas, who insisted that the stakes of the game of chance must be objectively determined. He noticed that if Daniel's hypothesis was applied to the problem, each player would pay different stake to Peter to enter the game while Peter's potential risk remained the same.

1.3 Solutions and impact on the development of Expected Utility Theory

The moral expectation theory was accepted by the majority of mathematicians of the late eighteenth and early nineteenth centuries. However, one of the notable opponents of it was Jean Le Rond d'Alembert (1717-1783). Regardless of the opposition, the theory gained even wider recognition with the publication of the monumental treatise of Laplace, who further developed Bernoulli's idea. Nevertheless, during the nineteenth century, the theory lost some of its initial attention and was only cursorily mentioned or even rejected by some French mathematicians such as Poisson, Bertrand or Poincare. What is more, Bertrand even satirised the theory by creating an imaginary dialogue between two of Bernouli's students.

Since the creation of the paradox, there have been numerous attempts to solve it. While until 1950' it had mainly attracted attention of mathematicians, later it became the subject of discussion among economists or even some philosophers and social scientists. Probably the most common solution was the idea of restricted game achieved by limiting the bankers capital, and as a result, reducing the number of possible tosses of a coin. Many scholars of that time who wrote on the St. Petersburg Paradox shared the view that this assumption made the game possible in the real world. However, D'Alembert raised objection to this solution. He was not the only opponent as Bertrand made two suggestions concerning potential insolvency of the banker. The former idea was to substitute the object of the game with even smaller objects of greater quantity i.e coins with grains of sand, grains of sand with hydrogen molecules and so on, in order to diminish the fear of banker's insolvency. The latter idea, later partially accepted by the famous economist John Maynard Keynes (1883-1946) and others, was to give a player a note confirming any potential debt in case of banker's insolvency. It is very intriguing and important to mention that the matter of the player's view on such a solution (and potential consent) was not mentioned. The matter was developed further by Paul Samuelson and Lloyd Stowell Shapley - both a Noble-prize winners in Economic Sciences. The latter was a great mathematician and economist, known for Shapley value and his contribution to the game theory, who passed away on the 12th of March 2016 at the age of 92.

Another approach to the paradox's solution was given by Buffon, who focused on disregarding the negligible probabilities (the initial idea regarding this solution was presented by Nicolas and Daniel Bernoulli and was mentioned earlier). In order to establish a proper threshold for neglecting the probabilities, Buffon presented an interesting example which justified his reasoning. He stated that according to mortality tables of his times, the odds of a healthy 56-year-old man dying within twenty-four hours were 1 to 10189. Hence, any event with a probability of occurrence 1/10000 or less might be disregarded. The argument of disregarding the probabilities was accepted by aforementioned D'Alembert, however, without specifying the threshold. Another famous mathematician Emile Borel (1871-1956) was also in favour of such a solution and in one of his books he determined some thresholds which, according to him, were appropriate in various scales - human, terrestrial, cosmic, etc.

Gabriel Cramer and Daniel Bernoulli's concave transformations of the winnings are considered the first "solution" of the St. Petersburg Paradox. As it was already mentioned, Cramer's suggestion was to apply square root in order to transform the winnings, while Daniel Bernoulli suggested applying the natural logarithm. Additionally, the latter also incorporated the assumption that the utility of winnings is inversely proportional to the player's wealth. As indicated by the equation number 1.10 on page 20 it is easy to notice that the larger the player's wealth is, the smaller his marginal utility of winnings is. Secondly, additional constant winnings have decreasing marginal utility and finally, the utility of gain falls short of the (absolute) disutility of an equivalent amount of money lost (Seidel, 2013). It is important to point out that all of the above assumptions are in accord with modern economics. The hypothesis was later proved by Weber (1834) and Fechner's (1860) experimental investigations on psychophysics. Furthermore, Leonard Jimmie Savage (1917–1971), the creator of Subjective Expected Utility Theory said that *"To this day, no other function has been suggested as a better prototype for Everyman's utility function"*. Concluding from all of the above, Daniel Bernoulli and aforementioned Buffon can be considered a precursors of Kahneman and Tversky's prospect Theory (Peterson, 2008).

Daniel Bernoulli was able to utilise the dependence of the marginal utility of winnings and losses on a person's wealth during his times. Not only was he able to explain why it was more profitable for some less wealthy persons to buy insurance for particular hazard and for richer people not to buy it, but also presented quite a modern theory of portfolio selection for risk spreading. Daniel Bernoulli's thesis concerning diminishing marginal utility of money has been immensely influencial since it serves as the basis for the standard theory of risk aversion, which explains a wide variety of economic phenomena (Joyce, 2011).

1.3.1 Menger's Super-Petersburg Paradox and its input to Von Neumann–Morgenstern utility theory

Cramer-Bernoulli solution of the St. Petersuburg Paradox did not pass the test of time. It was refuted by Carl Menger[9] (1840–1921) - an Austrian economist, who created a counterexample, later called Super-Petersburg Paradox by another economist Paul Anthony Samuelson[10] (1915-2009). Menger showed that applying "sufficiently concave" transformation of the winnings is only a sufficient but not at the same time necessary condition to solve the paradox.

Menger's idea was to replace the payout $a(n) = 2^n$ by $\hat{a}(n) = e^{2^n}$ which after applying Bernoulli's concave transformation $ln(\cdot)$ to $\hat{a}(n)$ regains the paradox. The same can be done to Cramer's solution by replacing $a(n) = 2^n$ with $\hat{a}(n) = (2^n)^2$. Hence, applying Cramer's square root transformation $\sqrt{\hat{a}(n)}$ the paradox is regained again. In general, Menger's counterexamples show that for each and every increasing and unbounded utility function an increasing transformation can be found such that the transformed winnings converge

[9]Carl Menger was born in the Polish city of Nowy Sącz which was at that time the teritory of Austrian Partition in Austrian Galicia. He is mainly known as the founder of the Austrian School of economics and his contribution to development of the theory of marginalism (marginal utility).

[10]Paul Samuelson was an American economist and statistician, a son of Jewish immigrants of Polish origin. He was the first American to win the Nobel Memorial Prize in Economic Sciences and the founder of faculty of economics at Massachusetts Institute of Technology.

relatively faster to infinity than the probabilities converge to zero. Carl Menger was the first person to formulate and prove the necessary and sufficient condition to prevent the occurrence of the St. Petersburg Paradox. The main contribution of Menger to the solution of the paradox was the necessary condition of utility function to be bounded. In other words, he showed that the St. Petersburg game has a finite solution only if the utility of winnings is bounded.

The aforementioned Paul Samuelson called Menger's breakthrough a *"quantum jump"* in the analysis of the St. Petersburg Paradox. The solution was also praised by other economists e.g. Kenneth Arrow (born in 1921) - American economist and the youngest winner in history of the Nobel Memorial Prize in Economics; and Robert John Aumann (born in 1930) - a member of the United States National Academy of Sciences and the Noble Prize winner in Economics for his work on conflict and cooperation through game-theory analysis.

It is most intriguing that the St. Petersburg Paradox had to wait such a long time until Menger's formulation of the necessary and sufficient condition to prevent its occurrence. According to Christian Seidl (Seidl, 2013) it was due to comprehension of utility and its development over several centuries. Many scientists of that time considered utility as something *"palpable, immutable, and interpersonally comparable"* (Dutka, 1988). It was not until 1906 when for the first time Italian economist Vilfredo Pareto in his *Manual* accepted the interpersonal noncomparability of utility. Paul Samuelson once concluded that "to the preceeding generation of economists, interindividual comparisons of utility were made almost without question; to a man like Edgeworth[11], steeped as he was in the Utilitarian tradition, individual utility-nay social utility-was as real as his morning jam.".

Menger's argument was first publicly presented in 1927 to the Economic Society of Vienna. His work did not receive much attention until 1934 when Oskar Mongernstern, who was at that time the managing editor of the *Zeitschrift fur Nationalokonomie*, decided to publish it in his journal. Menger's input was the most influential for the later creation of Expected Utility Theory by John von Neumann and Oskar Morgenstern. The proof of this fact is how Morgenstern himself recalls his cooperation with John von Neumann on the matter:

"...we decided that we would settle on thinking about numerical utility. It did not take us long to construct the axioms on which the present theory is based that gave us a firm utility concept, that of an expected utility, numerical up to a linear transformation. ...Regarding risk, Karl Menger's important paper of 1934 on the St. Petersburg Paradox ... played a great role. ...the construction of

[11]Francis Ysidro Edgeworth (1845–1926) was an Anglo-Irish philosopher and political economist. In 1891 he was appointed the founding editor of The Economic Journal

axioms of our expected utility came quite naturally. I recall vividly how Johnny rose from our table when he had set down the axioms and called out in astonishment: "But didn't anyone see that?". . . . It was largely my doing that this utility theory was developed . . . "

2 Expected Utility Theory

2.1 Introduction

In the first part of the following chapter Von Neumann–Morgenstern utility theory will be described in detail. Firstly, the process of creation, original arguments and presentation of Von Neumann–Morgenstern axioms will be discussed on the basis of their most famous book *Theory Of Games And Economic Behavior* (Von Neumann and Morgenstern, 1953). The topic will also be elaborated by the interpretation of the axioms. The second part of this chapter will be dedicated to the elementary notions of making decisions under certainty and uncertainty as the theory is most often presented in such context. The chapter will be concluded by several examples and formal definitions concerning risk propensity types in order to provide necessary notions for the last chapter of the thesis.

2.2 Von Neumann–Morgenstern utility theorem and axiomatization

John von Neumann and Oskar Morgenstern at the beginning of the third chapter of their most famous book *Theory Of Games And Economic Behavior* clearly state their goal concerning the utility. By the use of wide notion of utility they intend to describe the fundamental concept of individual preferences. In the often quoted passage they refer to the economists' reaction to their invention and the well-known concept of "indifference curves".

"Many economists will feel that we are assuming far too much, and that our standpoint is a retrogression from the more cautious modern technique of "indifference curves"."

I am now going to carry out a detailed analysis of this chapter which is considered one of the roots of the theory. Von Neumann and Morgenstern start their considerations with a point to treat utility as a numerically measurable quantity. Here, they make an interesting remark that in the literature of that time such concept was considered radical. In order to prove that this concept is not as radical as it was suggested, they present meticulous arguments supported by step-by-step reasoning.

They decided to start they argumentation by comparing the notion of utility to physical sciences. It resulted in a very intuitive and at the same time very vivid argument. The authors suggest that every measurement or even a claim of measurability must at the end be strictly connected with some sort of immediate sensation which should be considered as natural and not requiring the need of further analysis. In the context of physics, an example of such sensation might be light, heat or even muscular effort. One might begin to wonder what would be the immediate sensation connected with utility. The answer seems quite natural - the preference. The preference of one object over the other or the set of objects against another. So far this argument only lets us state when one utility for a person is greater than the other. What is particularly important is that it is not yet a basis for numerical comparison of utilities - neither for an individual nor for the comparison within the group of individuals. As we cannot intuitively think of any easy way to add two utilities for the same individual, it might suggest the non-numerical character of the utility. The mathematical procedure to describe such situation is the use of indifference curve analysis.

The point of all the above is that the situation is similar to the early beginnings of the theory of heat. Initially, it was clear on the intuitive level that one body feels warmer than the other, however, no-one was able to indicate by how much or how many times "warmer". Funny enough, this argument was given by Von Neumann and Morgenstern to show that the ultimate shape of new theories in the future is almost always impossible to forecast *a priori*. It is a common knowledge that heat can be quantitatively described not only by one number but two - the quantity of heat and temperature. Both of these characteristics are numerical, however only the quantity is additive. By the use of this argument, the authors might have wanted to influence the reader and make him more willing to accept their theory or at least to make him less critical about it. As the above example shows, one should be very cautious while negatively assessing new theories since he cannot be sure about their future development and ultimate appliance. In order to even further support the argument the example of development of the theory of light, colours and wave lengths was presented. All of these notions became numerical, however, in completely different formal sense. Recapitulating, the point of Von Neumann and Morgenstern was that even though at that time the notion of utility might have seemed unnumerical, the history of heat theory might repeat in the future. Hence, the theoretical considerations of the formal use of a numerical utility should not be abandoned.

2.2.1 Probability and numerical utilities

Following this idea, only little more effort than using the assumptions of indifference curve analysis is needed to achieve a numerical utility. First and foremost, the numerical utility requires the possibility to compare the differences in utilities. It is a bigger assumption than sole ability to state preferences. For further analysis, a few assumptions will be made. Firstly, let us assume that an individual has a surjective and complete system of preferences i.e. for any two objects (or imagined events) he has a unequivocally defined intuition of preference. In other words, when faced with two alternative events (possibilities) he is able to clearly state which one of these two he prefers. A natural extension of this assumption would be the possibility for the individual to compare not only single events but also combinations of events with attached probabilities. Such extension is needed for application to economy since many economic activities are explicitly dependent on probability - which is usually unknown or hard to estimate (the simplest example - insurance).

Let us assume the following situation. Let three events be denoted by A, B and C. For the sake of simplicity let the probability of occurrence of events B and C be equal to 50% i.e. the probability of B occurring is equal 50% and if B does not occur, than event C must occur with the remaining probability (which in this case is 50%). Two further assumptions are made considering this situation. Firstly, the two alternatives B and C are mutually exclusive so there is no possibility of complementarity. Secondly, we assume absolute certainty of the occurrence of either event B or C.

In our example we expect the individual to have a clear intuition whether he prefers event A to the $50-50$ combination of events B and C or the opposite (the combination of B and C to the event A). Having established the example, let us consider three cases. When the individual prefers event A to event B and at the same time event A to event C (using modern game theory nomenclature: $A \succ B \wedge A \succ C$), it is clear that he will also prefer event A to the combination of events B and C. Similarly, if he prefers event B to event A and at the same time event C to event A ($B \succ A \wedge C \succ A$), he will prefer the combination of events B and C to the event A. However, if he should prefer event A to let us say B but at the same time C to A ($A \succ B \wedge C \succ A$), than any statement of his preference of A to the combination of B and C in such case gives us a fundamentally new information. Hence, this case provides a base for numerical estimation of the fact that his preference of A over B is "greater" than his preference of C over A.

The above case can be explained by the use of a very simple example. Let us assume that an individual prefers a glass of tea to a cup of coffee and at the same time that he prefers a cup of coffee to a glass of milk. In order to get to know if the second preference (i.e. difference in utilities) is greater than the first one, it is enough to make him decide whether he prefers a cup of coffee to a glass which content will be determined by a toss of a fair coin (heads = tea, tails = milk).

It is important to point out that so far we have only outlined the intuition which enables an individual to decide which of the two "events" is preferable. So far we have not talked about the estimation of the relative sizes of preferences (differences in utilities). If the view presented above is accepted, it gives us a criterion how to compare the preferences of C over A with the preference of A over B. Hence, the differences of utilities become numerically measurable. The fact that such comparison is sufficient for a numerical measurement of "distances" was first observed in economics by Vilfredo Pareto. It is worth mentioning that exactly the same argument was used a lot earlier in mathematics by Euclid for the position of points on a line which actually was the very basis of his classical derivation of numerical distances.

The abovementioned example was extended further by the authors. They intended to show even more direct way of achieving numerical measures by the use of all possible probabilities. Let us again consider the three events A, B and C, however, this time with specified order of preference: the individual prefers event C to A and at the same time A to B. This gives the following order of preference written in modern nomenclature $C \succ A \succ B$. Now, let us introduce the new parameter α which is a real number from the interval 0 to 1: $\alpha \in \mathbb{R} \wedge \alpha \in < 0, 1 >$. Parameter α should be associated with the events in such a way that event A is equally desirable with the combined event consisting of event C with associated probability described by the parameter α and event B with associated remaining probability $1 - \alpha$. Having defined α, the authors suggest to use it as a numerical estimator of the ratio of two preferences - preference of event A over B to the preference of event C over B.

The case above enables us to present another example of appliance of the reasoning. Let us assume that we consider a certain good. Now we aim to determine the ratio of utility of having one unit of such good to the utility of having two units of it. Let us denote this ratio by q. We are ready to construct a query for the individual which will enable us to determine his preference by utilising the introduced ratio q. We give an individual a choice of having 1 unit of the considered good with certainty (analogous to the event A of the theoretical example given prior to this one) or playing for the chance of having 2 units with the probability α (analogous to event C) or ending up with nothing with remaining probability $1 - \alpha$ (analogous to event B). Hence, we can determine that if he prefers one certain unit than $q > \alpha$, if he prefers to play than $q < \alpha$ and finally if he cannot state his preference than $q = \alpha$.

Before we proceed further with our considerations, a few remarks are needed to be made. First and foremost, further analysis of the above example requires the use of the axiomatic method which will be introduced later on in this chapter. Furthermore, in order to avoid potential misunderstandings and for the sake of simplicity, when referring to any "events" we perceive them as future events. What is more, we intend to treat these events as occurring at exactly the same, standardised moment in the immediate future. However interesting it might be to consider events in different places in the future (for example in scope of the theory of saving and interest), it complicates the consideration far too much.

All of the above considerations are directly connected to the concept of probability. The authors wisely point out that the probability might be perceived as a subjective concept of estimation or as a frequency in long runs. Since we want to construct an individual and numerical estimation of utility the later view should be used. It is well founded and gives us the necessary numerical background for further analysis.

2.2.2 Principles of measurement

Going back to the main aspect of our considerations i.e. numerical measurement of the utility, we should point out that nowhere did we get any basis for neither qualitative nor quantitative comparison of utilities between different individuals. For the individual, this measurement procedure relies on the hypothesis of completeness in the system of his preferences. However, it is important to consider another case. Let us imagine a situation in which the individual cannot state his preference of one event over the other and at the same time he cannot clearly state that these events are equally desirable for him. This case seems realistic but unfortunately makes numerical measurement of the utility and the method of indifference curves impracticable. From the mathematical perspective this problem is connected with the theory of ordered sets. The problem comes down to the question whether considered events (with respect to preference) form only partially or completely ordered set.

By the above considerations, the authors wanted to show that the method of indifference curves in some cases might imply either too much or in the others too little. Two cases should be considered in order to explain this statement. Let us assume that the preferences of an individual are not all comparable. In such case we cannot construct indifference curves as it is required that any points on the same indifference curve must be identified. Hence, there is no place for incomparability. On the other hand however, when the individual's preferences are all comparable, than a uniquely defined numerical utility can be obtained which makes the indifference curves redundant.

Prior to the discussion about the measurability details, the authors again clearly justify the reason for developing their theory. They argue that even though one may wonder what is the reason for measuring individual's utility, since in real life no one seems to numerically measure for instance intensity of light, level of heat or muscular effort, these phenomena had to be measured in order to develop the science of physics. Hence, even though one may not perform such measurements on every day basis, he certainly uses the results of such measurements - directly or indirectly. On the ground of this argument, Von Neumann and Morgenstern claim that once the understanding of economic phenomena is developed further in the future, the need for such measurability might become necessary and therefore might affect the life of individuals.

Before discussing details of measurement principles, Von Neumann and Morgenstern wanted to assure the reader that the numerical scale of utility has not been forced anyhow. They go back to the example in which the individual preferred event A to the combination of events B and C (with equal 50% probability of occurrence) assuming the preference of event C to A and at the same time event A to B. While discussing this example earlier, the authors stated that it gives the basis for numerical estimation that individual's preference of event A over B exceeds the preference of event C over A. It must be emphasised that the authors did not intend at any point to assume or take for granted that one preference may exceed another. The only assumption they wanted to make was that this example provides a good empirical evidence that imagined events can be combined with probabilities and than utilities as well, regardless what they may be.

In the next part of the chapter, Von Neumann and Morgenstern present detailed principles of measurement by giving examples of measurements in the field of physics. It is often observed in science that some quantities which are not *a priori* mathematical are used to describe physical world. Some of these quantities can be grouped together in domains which are characterised by certain operations - well-defined, possible and natural for its domain. Let us give a few examples from the field of physics to properly illustrate our reasoning. For instance, addition can be considered as natural operation for physically defined quantity of "mass". The same can be said about physico-geometrically defined quantity of "distance". However, the quantity of "position" defined in the same way does not permit such operation. Nevertheless, it allows us to create the "center of gravity" of two positions.

Even more examples related to physics could be mentioned here, however, more important is to understand the notion of the mentioned "natural" operation in a given domain. In order to avoid misunderstanding, one should be aware that even though some operations in particular domains might be called as "addition" (more specifically "of additive nature") and resemble some common mathematical operations, it does not mean in any case that these two operations with the same name are identical. It only indicates that they have similar characteristics and we might hope that at the end some correspondence between them will be established. Von Neumann and Morgenstern in their book elaborate on this matter further by concerning other examples and analysing more complex mathematical transformations. The authors conclude their deliberation by stating that the situation of utility is very similar comparing to given examples. They remark that "utilities are numerical up to[12] a monotone transformation". This view is widely accepted in economy and supported by the use of indifference curves.

In order to narrow down the system of transformations, additional operations or relations should be found in our domain of utility. Vilfredo Pareto once suggested that it would be enough to introduce an equality relation for the consideration of utility. In our case it would result in reduction of our transformation system to the linear transformations. This view can be compared with what Euclid suggested for position on a line. The concept of "preference" can be compared to the Euclid's concept of "lying to the right of". Unfortunately, this suggestion cannot be used because, as Von Neumann and Morgenstern point out, this relation does not seem "natural" and what is more cannot be interpreted by reproductive observations.

Despite the negative assessment of the abovementioned relation, Von Neumann and Morgenstern believed that there is another "natural" relation which can bring the same result i.e. narrow the transformation system to the linear transformations. The authors took advantage of already introduced concept of two utilities with associated two alternative probabilities to them: α and $1 - \alpha$, given that $\alpha \in \mathbb{R} \wedge \alpha \in < 0, 1 >$. Due to the fact that a major resemblance to the formation of centres of gravity can be found in the mentioned process, the authors decided to use the similar terminology for the sake of simplicity.

2.2.3 Conceptual structure of the axioms

In the following section I will use the same notation as Von Neumann and Morgenstern used in order to enable the reader to make an easy comparison

[12] *"up to"* is a mathematical parlance. The phrase *"up to"* is used while discussing the elements of a set X and the condition(s) Y under which subsets of those elements may be considered equivalent. In other words, when it is said that "the elements x and y of the set X are equivalent up to Y" it means that we consider elements x and y as equivalent if criterion Y is ignored. A typical example of the use of such expression in mathematics is given while discussing the solutions to an indefinite integral. The solution to an indefinite integral $\int f(x)dx$ is a function $g(x)$ up to addition by a constant i.e. $\int f(x)dx = g(x) + C$.

with the original considerations.

Let u and v denote utilities. Our "natural" relation connected with those utilities is preference. Therefore, if we assume that u is preferable to v, than we would denote it as follows: $u > v$. [13] The "natural" operation, however, will be denoted: $\alpha u + (1-\alpha)v$, where again $\alpha \in \mathbb{R} \wedge \alpha \in < 0, 1 >$. It can be interpreted similarly to the physical interpretation of centre of gravity i.e "centre of gravity of u, v with the respective weights α, $1-\alpha$", or more directly connected to our reasoning i.e "combination of u, v with the alternative probabilities α, $1-\alpha$". Now, if we agree to above assumptions, the mathematical (numerical) concept must be found which will convey both, the relation $u > v$ and the operation $\alpha u + (1-\alpha)v$ for the utilities.

Let us start by denoting the following correspondence:

$$u \longrightarrow \rho = \mathrm{v}(u),$$

where u denotes the utility and $\mathrm{v}(u)$ denotes the number which is attached to the utility u by the above correspondence. Having established the correspondence, we now state the following requirements:

$$u > v \Rightarrow \mathrm{v}(u) > \mathrm{v}(v) \tag{2.1a}$$

$$\mathrm{v}(\alpha u + (1-\alpha)v) = \alpha\mathrm{v}(u) + (1-\alpha)\mathrm{v}(v) \tag{2.1b}$$

Let us create two correspondences denoted as follows:

$$u \longrightarrow \rho = \mathrm{v}(u) \tag{2.2a}$$

$$u \longrightarrow \rho' = \mathrm{v}'(u) \tag{2.2b}$$

They set up a correspondence between numbers

$$\rho \leftrightarrows \rho', \tag{2.3}$$

which may be also represented by a function:

$$\rho' = \phi(p) \tag{2.4}$$

From the fact that our two correspondences i.e 2.2a, 2.2b fulfil the requirements 2.1a, 2.1b, we can conclude that the function $\phi(p)$ from equation 2.4 must preserve the analogical relation $\rho > \sigma$ (please note that now ρ and σ represent

[13] Von Neumann and Morgenstern used symbol $>$ instead of $\succ$ which is commonly used nowadays. For the sake of coherence, I will preserve authors original nomenclature and use the symbol $>$ and $<$ for indicating preference in this section.

numbers!) and the associated operation $\alpha\rho + (1-\alpha)\sigma$. When we look closer into above considerations, we can contrast both sides of equations in each case. By this we mean, that each left-hand side preserves the "natural" concepts for utilities which were discussed before and each right-hand side represents known and intuitive concepts for numbers. Hence, we can now represent this reasoning in an analogous way:

$$\rho > \sigma \Rightarrow \phi(\rho) > \phi(\sigma) \tag{2.5a}$$

$$\phi(\alpha\rho + (1-\alpha)\sigma) = \alpha\phi(\rho) + (1-\alpha)\phi(\sigma) \tag{2.5b}$$

The above equations imply that $\phi(\rho)$ must be a linear function. In formal mathematical language we can denote it as follows:

$$\rho' = \phi(\rho) \equiv \omega_0\rho + \omega_1, \tag{2.6}$$

where ω_0, ω_1 are fixed numbers (constants) and additionally $\omega_0 > 0$.
From all the above, we can conclude that if a numerical representation of utility in a form of a correspondence described in 2.2a fulfilling both the requirements 2.1a and 2.1b exist at all, it must be in a form of linear transformation as shown in 2.6. Hence, utility must be a number up to linear transformation.

Prior to defining the axioms, Von Neumann and Morgenstern admit in a sense that their axioms might be vulnerable. They state that the choice of the axioms is rather subjective and never fully objective. What is more, they are usually created in order to achieve certain goal. Indeed, when established it should be natural to derive from them certain theorems and to this extent they might be considered exact and objective. Having secured the statement that there is always a subjective and objective side of the process, the authors proceed to list in their opinion the most important features (requirements) of the axioms.

- there should be a relatively small number of axioms
- the system of axioms should be as simple and transparent as possible
- each axiom should have an immediate intuitive meaning by which its appropriateness may be judged directly

The authors point out that the third requirement, despite its vagueness, is of the most importance because their aim is to create an intuitive concept which would be possible to be treated in a mathematical way and hence it would be possible to observe what hypothesis it requires. When it comes to the objective side of creating the axioms, from the previous considerations we want them to imply the existence of correspondence 2.2a satisfying the requirements 2.1a and 2.1b.

It is essential to point out that above considerations do not indicate in any way how to find an axiomatic treatment. Von Neumann and Morgenstern formulated a set of axioms which seemed to them "essentially satisfactory".

2.2.4 Original version of the axioms

Let U be a system of entities (abstract utilities) $u, v, w, \ldots$. For any number $\alpha \in \mathbb{R} \wedge \alpha \in < 0, 1 >$, a relation $u > v$ and an operation $\alpha u + (1-\alpha)v = w$ is defined. The above concepts satisfy the following axioms:

1. **COMPLETENESS**

$$u > v \text{ is a complete ordering of } U \tag{2.7}$$

For any two u, v one and only one of the three following relations holds:

$$u = v, \qquad u > v, \qquad u < v \tag{2.8}$$

2. **TRANSITIVITY**

$$u > v \wedge v > w \Rightarrow u > w \tag{2.9}$$

Ordering and combining. Let $\alpha, \beta, \gamma \in (0, 1)$:

$$u < v \Rightarrow u < \alpha u + (1-\alpha)v \tag{2.10a}$$

$$u > v \Rightarrow u > \alpha u + (1-\alpha)v \tag{2.10b}$$

3. **CONTINUITY**[14]

$$u < w < v \Rightarrow \exists\alpha \;\; \alpha u + (1-\alpha)v < w \tag{2.11a}$$

$$u > w > v \Rightarrow \exists\alpha \;\; \alpha u + (1-\alpha)v > w \tag{2.11b}$$

4. **INDEPENDENCE**

Algebra of combining.

$$\alpha u + (1-\alpha)v = (1-\alpha)v + \alpha u \tag{2.12a}$$

$$\alpha(\beta u + (1-\beta)v) + (1-\alpha)v = \gamma u + (1-\gamma)v \;\; where \;\; \gamma = \alpha\beta \tag{2.12b}$$

From the above axioms it is possible to observe that they imply that the correspondence 2.2a (page 32) satisfying the requirements 2.1a and 2.1b does exist. Therefore, we can state that the conclusion about linear transformation

[14] The symbol $\exists$ denotes the existential logic quantifier which stands for "there exists" or "exists".

holds good as the abovementioned system of abstract utilities U is numerical up to a linear transformation. The formal construction of such correspondence by the use of the axioms can be considered as a purely mathematical process. Although the process itself is quite long, it should not cause substantial difficulties as it uses common and conventional mathematical methods.

2.2.5 Interpretation of the axioms

In the following section, each of the axioms will be briefly discussed giving explanation together with authors' reason for development.

1. COMPLETENESS

Completeness has been partially discussed prior to axioms introduction. This is a basic mathematical concept which is typical for discussing utilities and preferences. It is also applied while using the method of indifference curve analysis. As additional remark, we should point out that the system of utilities U is completely ordered. That means that for our defined relation $u > v$ we can write $u < v$ when $v > u$. Most importantly, however, we should notice that for any two selected utilities u and v there are only three possible relations: $u = v$, $u > v$ and $u < v$ (see 2.7 and 2.8 on page 34). These directly implies the transitivity.

2. TRANSITIVITY

Transitivity is another generally accepted and common mathematical concept. When we consider three utilities u, v and w and clearly state two relations of preference between them i.e. $u > v$ - unambiguous preference of u over v; and $v > w$ - unambiguous preference of v over w, than from these two assumptions we logically imply the third relation i.e. $u > w$ (see 2.9 on page 34). In other words, it requires no more than common sense to deduce that if an individual prefers u to v and at the same time v to w, he will with all certainty prefer u to w when given such choice.

The next implication (see 2.10a) means that if we assume preference of v over u ($v > u$) than even a chance of achieving v with probability $1 - \alpha$ alternatively to u is still preferable. We can assume so, due to the fact that we have already ruled out any possibility of complementarity in our earlier considerations. Implication 2.10b is analogous to the implication 2.10a with the difference of putting "less preferable" in place of "preferable".

3. CONTINUITY

Let us consider the formula 2.11a. Let u, w and v denote the utilities with two established relations such that together they form an unambiguous order of preference i.e. $u < w < v$ (we assume preference of w over u (first relation) and at the same time preference of v over w (second relation)). From these assumptions we can imply the **existence** of parameter α[15] such that $\alpha u + (1-\alpha)v < w$. It should be interpreted as follows: the combination of u and v with probabilities α and $1 - \alpha$ respectively, will not affect anyhow the preferability of w to such combination, given that the chance $1 - \alpha$ is small enough. In other words, however desirable v may be, it is possible to reduce its influence as much as desired by associating with it sufficiently small chance. This assumption was named by the authors "continuity". The next formula 2.11b is again analogous to the explained formula 2.11a in a similar way as in the previous axiom (2.10a and 2.10b).

4. INDEPENDENCE

Equation number 2.12a is sometimes referred to as an "additional" axiom of ordering in modern literature. Let u and v denote two utilities. Than equation 2.11a states that it does not matter in what order we name given utilities. Hence the name "ordering". It is a legitimate assumption as the mentioned utilities are considered alternative events (see explanation of transitivity axiom above and consideration of complementarity).

The following equation 2.12b states that it does not matter if the combination of two utilities (in this case u and v) is achieved in two consecutive steps - firstly by applying probabilities with component α i.e. α and $1 - \alpha$ and than probabilities with component β i.e. β and $1 - \beta$ - or in single step by applying probabilities with the component γ i.e. γ and $1 - \gamma$ where $\gamma = \alpha\beta$.

2.2.6 Final remarks concerning the axioms

Having established and explained the axioms, Von Neumann and Morgenstern again justify their invention and make some final remarks concerning potential problems with their theory. They start by asking if their axioms do not show too much. Directly from the axioms one can show the numerical character of utility in accordance with previous assumptions i.e. 2.2a on page 32 and its related requirements i.e. 2.1a and 2.1b. Furthermore, the second requirement 2.1b directly implies that numerical utility can be combined with probabilities just like mathematical expectations. However, we already know from the previous chapter that the concept of mathematical expectation has been questioned many times. The problem arises. Is it possible that there exists a utility (positive or negative) of gambling ("taking a chance")? Please notice that the use of

[15]we do not imply anything more about parameter α than its mere existence

mathematical expectation obliterates such utility. The authors ask a question how their axioms deal with this problem?

They clearly state that their axioms do not try to avoid it. They argue that even the postulate 2.12b which is connected to the idea of utility of gambling the most, is valid under the modern system of psychology applied to economics (Berka, 1976). Furthermore, the authors claim that their invention i.e. definition of numerical utility is suitable for the use with the mathematical expectations. They support this view by referring to the already discussed matter of "moral expectation" used by Daniel Bernoulli. Let us remind the reader that Bernoulli's concept was a suggestion on how to solve the St. Petersburg Paradox (using "moral expectation" instead of the mathematical expectation). What he did essentially was defining utility numerically as a logarithmic function of individual's wealth. However, Von Neumann and Morgenstern admit at the end that due to the characteristics of construction of their theory, it is inevitable to face some contradictions while trying to analyse concepts like "utility of gambling". Moreover, they confidently state that anyone who has ever tried to make axiomatisation of this matter would certainly agree with the point that some minor contradictions cannot be completely omitted.

Finally, let us restate the most important aspects of the theory. First and foremost it should be clearly pointed out that all of the above considerations apply only to the utilities perceived by single individual. Any of these considerations do not imply at any point anything about potential comparison of utilities between more individuals. Secondly, there is a lot more to be said regarding methods which utilise mathematical expectation. Although the authors were aware of the existence of many interesting questions regarding this matter, their goal to lay the ground for further analysis by developing essential notions and axioms was achieved. Recapitulating, Von Neumann and Morgenstern developed simple axioms which are valid for the relation $u > v$ and the operation $\alpha u + (1 - \alpha)v$ which they established. By these means they managed to numerically define utility up to a linear transformation.

2.3 Decision theory - decision making under certainty and uncertainty

Having discussed the notions of mathematical expectation and expected utility theory, in the following part of the thesis I would like to focus more on their appliance. The book *Theory Of Games And Economic Behavior* by John von Neumann and Oskar Morgenstern is considered by many as a "bible" of Game Theory. This publication was without a doubt a milestone in development of this branch of mathematics. In my opinion, Expected Utility Theory together with development of Game Theory made an extraordinary "bridge" between mathematics and economics. Not only did it change the perspective of mathematicians on some economic phenomena but also enabled economists to utilise mathematical apparatus in wider aspect of economic problems. One of the direct implications of Expected Utility Theory is the theory of making decisions under uncertainty or what we might call in a more economic way - modelling rational behaviour. This is probably one of the most common and basic appliances of the theory, yet very interesting (Straffin, 2011). In order to introduce basic concepts concerning the subject, a few formal definitions must be established first.

2.3.1 Decision making under certainty

Let us start with a simple example. Assume that the individual has 1000€ to invest. He has two choices. The first one is to invest the whole amount by making deposit account with guaranteed return of 4% in his bank. The second is to invest in investment portfolio offered by his financial advisor with return defined as follows: 25% chance of getting return of 1%, 25% chance of getting return of 3%, 25% chance of getting return of 4% and lastly 25% chance of getting return of 9%. Where should the individual decide to invest him money in? The answer to such question is not straightforward. One would say that it suffice to calculate the expected return (we may also call it EMV which stands for expected monetary value) of both options and select the one which offers higher one. It might not be visible at first sight but investing in portfolio has higher EMV. Let a denote making deposit account and b investing in portfolio. Than, it is very easy to calculate that:

$$EMV(a) = 4\%$$

$$EMV(b) = 25\% \times 1\% + 25\% \times 3\% + 25\% \times 4\% + 25\% \times 9\% = 4,25\%$$

Now we can see without a doubt that $EMV(a) < EMV(b)$. Does it mean that the individual should choose to invest in portfolio? Not necessarily. First and foremost it depends on individual's propensity to risk taking. It will be the main subject of the following considerations. Before moving forward let us remark that the expected return (or EMV) in terms of probability theory is no different from expected value of the discrete random variable which gives us different possible values of portfolio returns.

The above example was very conservative, let us analyse another one, this time more radical. Assume the individual is given a choice to receive 5€ for certain or to take a chance (gamble): 50% chance of receiving nothing (0€) and 50% chance of receiving 15€. Now imagine the same example but with significantly different amounts. 5 mln € for certain or a 50-50 gamble: nothing or 15 mln €. It is safe to assume that when faced with the first version of this example, most people would probably take a chance to receive 15 €, however, it is highly unlikely that more than few percent of people would actually take a risk for 15 mln € but would rather take guaranteed 5 mln € instead (given that they are not millionaires). One might ask what is the reason for such behaviour? Are people being inconsistent or irrational in this example? We have partially given example to this question already in the previous chapter. This discrepancy between the two version of the second example can be explained by diminishing marginal utility - the concept which was already suggested by Bernoulli. People do not value money purely by its amount but by the utility they might harvest from it. Connecting this two notions i.e. money utility and its diminishing marginal return, we can conclude that 15 mln € even though being three times bigger than 5 mln € does not bring three times more utility. Hence, when faced with the opportunity to get life-changing money with 100% certainty, they do not wish to risk it for a chance to get three times more or end up with nothing. So far, some analogies to what has been already discussed within this work might be noticed.

One question is still to be answered though. How can we define a rational person? The following considerations aim to present a very simple model of modelling rational behaviour which utilises some notions connected to expected utility theory of John von Neumann and Oskar Morgenstern. First of all we claim that every rational individual has his unique ranking of outcomes which is based on his personal preferences. Furthermore, what is in line with Von Neumann and Morgenstern's logic, we do not demand this ranking to be related anyhow to other individuals, however, we require it to be internally consistent. Hence, we can define a rational individual as having consistent preferences regarding outcomes and pursuing the most preferred one. We can illustrate this reasoning with a very simple example. Let us assume that a set of two possible outcomes is given - having a coffee or having a tea. When we ask a group of individuals about presented outcomes, they would surely express their preferences[16]. It is obvious that some of them would prefer having a coffee to having a tea and vice versa. However, when given a free choice each individual should choose to have a desired beverage according to his individual preferences. Otherwise he would be acting irrational i.e. for example preferring a tea and choosing coffee.

Having presented our logic and examples supporting it, we can now

[16] For the moment let us assume that there is no uncertainty connected with a choice i.e. making a choice will definitely lead to associated outcome.

formulate necessary definitions. They will consist of contemporary nomenclature a bit different but analogical to the one used by Von Neumann and Morgenstern. For all following considerations let Ω denote the set of possible outcomes where $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots\}$.

Definition 2 (Relations between preferences)
Let preferences of an individual be denoted in the following way:

- $\omega_1 \succ \omega_2$ *when the individual strictly prefers outcome* ω_1 *to the outcome* ω_2
- $\omega_1 \sim \omega_2$ *when the individual is indifferent about both outcomes*
- $\omega_1 \succeq \omega_2$ *(weak preference) when the individual either prefers* ω_1 *to* ω_2 *or is indifferent about these two outcomes*

With set up notation of preferences we can define rationality under certainty. In the following definition we an see close similarity to the axioms defined by Von Neumann and Morgenstern.

Definition 3 (Rationality under certainty)
We call an individual rational under certainty if his preferences satisfy the following conditions:

1. ***COMPLETENESS*** $\omega_1 \succeq \omega_2 \ \veebar \ \omega_2 \succeq \omega_1$ [17]
2. ***TRANSITIVITY*** $(\omega_1 \succeq \omega_2 \ \wedge \ \omega_2 \succeq \omega_3) \Rightarrow \omega_1 \succeq \omega_3$

Similarly to Von Neumann and Morgenstern axioms, the completeness requirement enables the comparability of the outcomes. The transitivity requirement, however, enables these outcomes to be ranked in order of preference (due to the weak preference, "ties" between some outcomes are allowed). Thanks to these two definitions we can now define the utility function also known as Von Neumann-Morgenstern utility function. Let us remark that an outcome ω might be of numeric nature (e.g. amount of money, number of winning votes etc.) or intangible (e.g. level of happiness, health condition etc.). The utility function, as we already know from Von Neumann and Morgenstern conclusions, assigns a numerical value to an outcome regardless of its nature (numerical or intangible). Moreover, we require this function to convey all necessary information (regarding an outcome) which are important to the individual to whom this function might belong. Now we are ready to formally define the utility function.

[17] sign $\veebar$ denotes exclusive disjunction (either X or Y)

Definition 4 (Utility Function)
A utility function is a function $u : \Omega \to \mathbb{R}$ which satisfies the following conditions:

$$u(\omega_1) > u(\omega_2) \iff \omega_1 \succ \omega_2$$

$$u(\omega_1) = u(\omega_2) \iff \omega_1 \sim \omega_2$$

The above definition directly implies that every individual who is rational under certainty should try to maximise his utility (Webb, 2007). If we consider payoff function π and its relation to the utility function, we can notice that for a particular choice a and the outcome associated with this choice $\omega(a)$, a payoff function has the following form: $\pi(a) = u(\omega(a))$.

2.3.2 Decision making under uncertainty

It should be noted that so far we have analysed only the situation where we consider choices which result in certain (known, guaranteed) outcomes. In order to follow the logic of Von Neumann and Morgenstern, we are now going to analyse situations in which outcomes are uncertain i.e. they depend on certain probabilities. This requires a definition of a lottery and a distinction between simple and compound lotteries.

Definition 5 (Lottery)

1. *SIMPLE LOTTERY: A simple lottery, which we denote by λ, is a set of probabilities of occurrence associated with every outcome $\omega \in \Omega$.*
   The probability of outcome ω occurring in the lottery λ shall be denoted as follows: $p(\omega|\lambda)$. Let Λ denote a set of all possible lotteries, than
2. *COMPOUND LOTTERY: a compound lottery is a linear combination of simple lotteries which belong to the same set Λ.*
   An example of a compound lottery: $q\lambda_1 + (1-q)\lambda_2$, where $q \in <0, 1>$.

By looking at the definition of compound lottery it is easy to notice the resemblance to the "natural" operation defined by Von Neumann and Morgenstern. Moreover, it is important to notice that we can regard compound lottery as a lottery in which the outcomes are lotteries as well. This remark is helpful for understanding the definition of rationality under uncertainty which we are now ready to introduce.

Definition 6 (Rationality under uncertainty)
We call an individual rational under uncertainty (or simply rational) if his preferences regarding lotteries satisfy the following conditions:

1. ***COMPLETENESS*** $\quad \lambda_1 \succeq \lambda_2 \;\veebar\; \lambda_2 \succeq \lambda_1$

2. ***TRANSITIVITY*** $(\lambda_1 \succeq \lambda_2 \ \wedge \ \lambda_2 \succeq \lambda_3) \Rightarrow \lambda_1 \succeq \lambda_3$

3. ***MONOTONICITY*** $(\lambda_1 \succ \lambda_2 \ \wedge \ q_1 > q_2) \Rightarrow q_1\lambda_1 + (1-q_1)\lambda_2 \succ q_2\lambda_1 + (1-q_2)\lambda_2$

4. ***CONTINUITY*** $(\lambda_1 \succeq \lambda_2 \ \wedge \ \lambda_2 \succeq \lambda_3) \Rightarrow \exists q \ \ \lambda_2 \sim q\lambda_1 + (1-q)\lambda_3$

5. ***INDEPENDENCE*** $\lambda_1 \succ \lambda_2 \Rightarrow q\lambda_1 + (1-q)\lambda_3 \succ q\lambda_2 + (1-q)\lambda_3$

Now we can see in full scope, that the above requirements were greatly influenced by the axioms developed by Von Neumann and Morgenstern. Taking into account the difference in creating above definitions i.e. introducing relation of weak preference, the requirements of completeness, transitivity and continuity are completely in line with Von Neumann and Morgenstern axioms. The first two requirements (completeness and transitivity) have exactly the same interpretation as in the case of rationality under certainty. The monotonicity together with continuity requirement can be interpreted analogically to the original axioms. Independence, however, requires more attention as it will be a subject of important considerations in the next chapter of this thesis. The independence requirement implies that preferences depend solely on the differences between lotteries (Aliprantis and Chakrabarti, 1998). In other words, it is possible to ignore components that are the same. As all the requirements were briefly commented, it is possible now to introduce one of the most important theorems concerning the matter of modelling rational behaviour which is the Expected Utility Theorem.

Theorem 1 (Expected Utility Theorem)
Let an individual be rational in the sense of the Definition 6 and have a utility function $u : \Omega \to \mathbb{R}$. Than, a rational individual acts in a way that maximises the expected utility function (the payoff function) $\pi(a)$ given by the following equation:

$$\pi(a) = \sum_{\omega \in \Omega} p\left(\omega | \lambda(a)\right) u(\omega) \tag{2.13}$$

Before discussing the subject further, let us present and briefly analyse another example. Let us assume that individual A has a utility function $u_A(\omega) = \omega$ while an individual B has a utility function $u_B(\omega) = \sqrt{\omega}$. Two lotteries denoted by L_1 and L_2 are presented to both individuals. The rules of the lottery L_1 are as follows: in order to take part in a lottery, an individual has to pay a fee equal to 300 € to have a chance to win 525 € with probability 50% or 325 € with the remaining probability (also 50%). The rules of the second lottery L_2 are as follows: in order to take part in a lottery, an individual has to pay a fee equal to 500 € to have a chance to win 644 € with probability 50% or 600 € with the remaining probability (also 50%). Which of the two lotteries is preferred by each individual?

Let us calculate the expected utility for both lotteries from the perspective of both individuals. Expected utilities of individual A for both lotteries are calculated in the following way:

$$L_1^A : \; 0,5 \times (525 - 300) + 0,5 \times (325 - 300) = 0,5 \times 225 + 0,5 \times 25 = 125$$

$$L_2^A : \; 0,5 \times (644 - 500) + 0,5 \times (600 - 500) = 0,5 \times 144 + 0,5 \times 100 = 122$$

while expected utilities of individual B for the same lotteries are:

$$L_1^B : \; 0,5\times\sqrt{525 - 300}+0,5\times\sqrt{325 - 300} = 0,5\times\left(\sqrt{225} + \sqrt{25}\right) = 0,5\times(15+5) = 10$$

$$L_2^B : \; 0,5\times\sqrt{644 - 500}+0,5\times\sqrt{600 - 500} = 0,5\times\left(\sqrt{144} + \sqrt{100}\right) = 0,5\times(12+10) = 11$$

Let us summarise the results in a form of a table. We can clearly see in the

	lottery L_1	lottery L_2
individual A	$L_1^A = 125$	$L_2^A = 122$
individual B	$L_1^B = 10$	$L_2^B = 11$

Table 2: Example 2. Source: *Own compilation.*

table 2 that even though two individuals are faced with exactly the same lotteries, according to their expected utilities they would choose differently. The individual A would chose lottery L_1 as $L_1^A > L_2^A$, however, the individual B would choose lottery L_2 as $L_2^B > L_1^B$ (even though lottery L_1 offers bigger maximum payoff i.e. 225). This example shows very important characteristic of making decisions under uncertainty. It shows that different individuals might perceive risk differently (or in some cases we might talk about different tolerance for risk). In the above example, we can say that an individual B is more cautious (risk averse).

Having shown by the above example that individuals might represent different attitudes towards risk, we can now present the formal definitions of risk propensity types in the theory of decision making.

Three types of individuals concerning their propensity to risk can be defined (Webb, 2007); those who are:

- **RISK AVERSE** - they prefer sure thing to gambles with the same expected value
- **RISK NEUTRAL** - they rank gambles according to their expected value
- **RISK SEEKING**[18] - they prefer the gamble to a sure thing with the same expected value (opposite to RISK AVERSE individuals)

[18]sometimes also called RISK PRONE

These can be defined in more formal way by using the utility functions.

Definition 7 (Risk taking propensities)
Let $u : [0, \infty) \to \mathbb{R}$ be the utility function. We define three types of individual's propensity to risk taking by the form of his utility function. If one's utility function is:

- ***linear** i.e. of the form: $u(\omega) = a\omega + b$, then an individual is **RISK NEUTRAL***
- ***strictly concave**, then an individual is **RISK AVERSE***
- ***strictly convex**, then an individual is **RISK SEEKING***

Usually for the purpose of simple modelling of rational behaviour, the utility functions are derived strictly from the above definition i.e. an example of typical risk averse utility function is $u(\omega) = \sqrt[n]{\omega}$, while for risk seeking utility function is $u(\omega) = \omega^n$, where $n \in \mathbb{N}$. Visual examples of utility functions and risk propensities associated with them can be found in the figure 1 below.

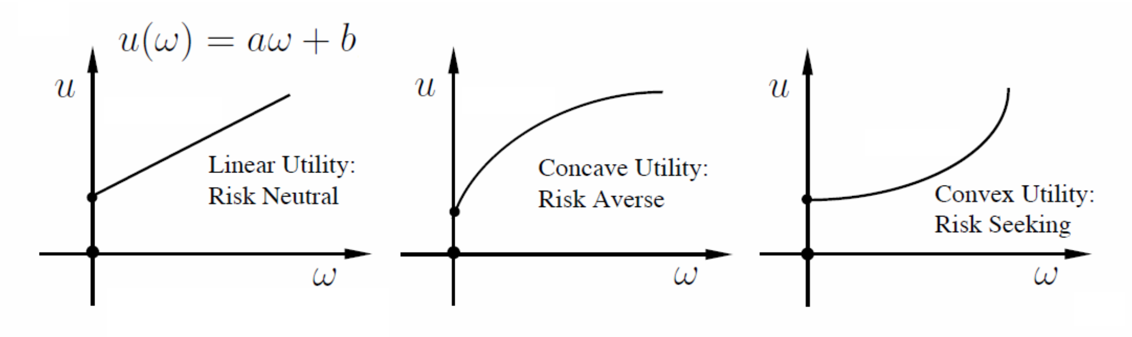

Figure 1: Examples of utility functions with indicated risk taking propensities. Source: (Aliprantis and Chakrabarti, 1998)

There also exists a bit more complicated definition of risk propensity utilising the concept of expected value.

Definition 8 (Risk taking propensities - $\mathbb{E}(\cdot)$)
Let $u : [0, \infty) \to \mathbb{R}$ be the utility function. An individual is said to be:

- *RISK NEUTRAL* *if* $\mathbb{E}(u(\omega)) = u(\mathbb{E}(\omega))$
- *RISK AVERSE* *if* $\mathbb{E}(u(\omega)) < u(\mathbb{E}(\omega))$
- *RISK SEEKING* *if* $\mathbb{E}(u(\omega)) > u(\mathbb{E}(\omega))$

given that $\mathbb{E}(\omega)$ exists and can be defined.

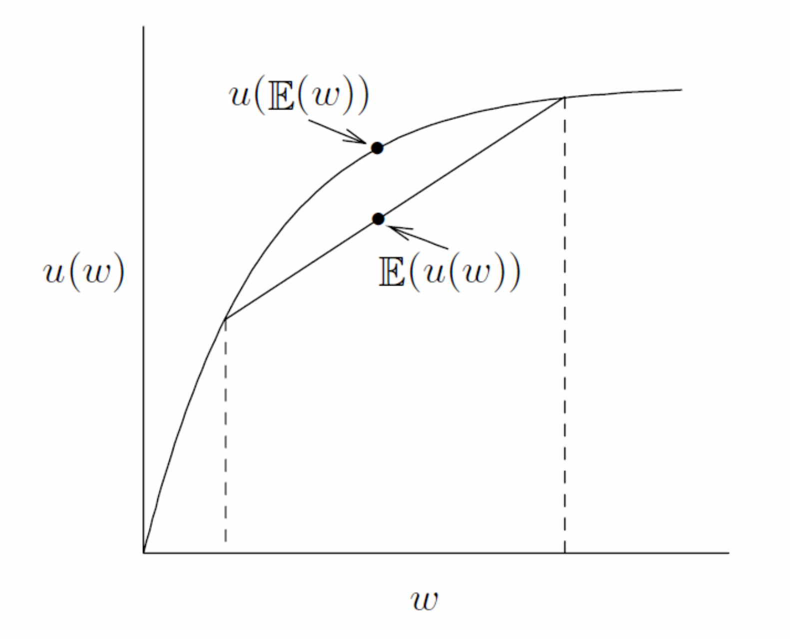

Figure 2: Risk taking propensities utilising $\mathbb{E}(\cdot)$. Source: (Webb, 2007)

The above definition can be checked by applying it to the second example on page 42. For L_1^B:

$$\mathbb{E}(u(\omega)) = 0,5 \times \left(\sqrt{225} + \sqrt{25}\right) = 10$$

$$u(\mathbb{E}(\omega)) = \sqrt{0,5 \times (225 + 25)} = 5\sqrt{5} \approx 11,18$$

because $u(\mathbb{E}(\omega)) > \mathbb{E}(u(\omega))$, on the ground of definition 8, we can confirm our previous conclusion that an individual B has risk averse utility function.

3 Maurice Allais's critique and Allais Paradox

3.1 Introduction

The following chapter will focus on Allais Paradox which is the most recognised violation of the Expected Utility Theory. The chapter will be divided into three parts - the first concerning Maurice Allais's critique of the Expected Utility Theory, the highlights of his arguments presented in his well known article *"Le Comportement de l'Homme Rationnel devant le Risque: Critique des Postulats et Axiomes de l'Ecole Americaine"* (Allais, 1953) and the origins of the Allais Paradox; the second part will introduce the necessary notions regarding indifference curves theory and the paradox itself; and finally, the third part will present some factors from psychology which also explain why some axioms of the Expected Utility Theory are violated.

3.2 Allais's critique of the American school and the origin of Allais Paradox

In the beginning of 1970's, the theory of decision making under uncertainty seemed to be a success in the field of economic analysis. One of the reasons for such a believe was the fact that it was created on solid axiomatic foundations (Karni, 2014). Furthermore, it significantly influenced risk analysis and started to be applied in real economic issues. Some of the economists of that time even believed that the theory would lay a groundwork for the upcoming 'information revolution' in economics. Nowadays it is known that the theory has been challenged on several grounds many times, both from the perspective of economics and other sciences (Hagen and Wenstop, 1984) (Broome, 1985). One of the first problems the theory of decision making under uncertainty had to face was the inconsistency of human propensity to risk taking in real life (Pope, 1986).

It is easy to observe that human attitude towards risk is far from being consistent. For example, there are people who are willing to pay for a chance to take risk while being aware that the expected payoff of such risky event is, on the average, lower that the actual stake paid for taking the risk itself. In other words, we are thinking about people who buy lottery tickets, are involved in betting on various games or simply try their luck in casinos. We already know from the previous chapter that such individuals in economic terminology are called risk seekers (or equivalently - of risk-seeking attitude). On the other hand, we can think of people of the opposite nature. There are people who pay in order to minimise, or even get rid of the risk completely, even though at the same time they are aware of the fact that the expected value of uncertain loss is, on the average, lower that the amount paid for insuring the event. Such individual usually buy insurance policies of various kinds e.g. car, health or life insurance policies. About such individuals we say that they are risk-averse.

However, it is crucial to realise that in real life it is impossible to find an individual who is either exclusively risk-seeking or exclusively risk-averse. Usually individual's propensity to risk taking differs depending on situation. Sometimes, the individual's attitude towards the particular situation might be different in different time scope e.g. different part of a day, different season of the year and so on. Professor Aswath Damodaran from the University of New York has given a nice real life example on that matter: 'The same person who puts his life at risk climbing mountains may refuse to drive a car without his seat belt on or to invest in stocks, because he considers them to be too risky.'.

Despite the difficulties in proving whether people are predominantly risk-seeking or risk-averse, some attitudes have been accepted in the economic literature regarding decision making under uncertainty. The most influential notion raised by Bernoulli was the diminishing marginal utility of wealth. Due to the fact that it is in line with the law of diminishing marginal utility applied to every commodity in microeconomic analysis of consumer behaviour, risk-aversion has been considered the norm in analysing decision making under uncertainty. Analogous to risk-aversion, risk-seeking was considered as an exception. In other words, an individual who seeks to maximise his utility must be risk-averse as he would not consider taking part in a fair game which expected payoff is lower or equal to the fee to enter it (he would only play games which expected payoff is greater than its price). It is due to the fact that diminishing marginal utility of wealth implies that disutility derived from a dollar's loss is always greater than the utility derived from a dollar's gain.

As it was already discussed in the previous chapter, John von Neumann and Oskar Morgenstern built their Expected Utility Theory basing on the concept of diminishing marginal utility of wealth. Expected Utility Theory uses a single-parameter criterion to evaluate possible choices with certain probabilities attached to them. Utility function attaches subjective utility value to each payoff. Finally, by comparing the value of expected utility of each choice one can rank them from the ones having the highest value to the ones having the lowest. Expected Utility Theory became popular as an analytical tool mainly due to the fact that the notion of diminishing marginal utility of wealth could be easily reflected by the use of increasing concave utility function. However, it is important to point out that the degree of risk aversion cannot be explicitly assessed from the curvature of the utility function.

Although von Neumann - Morgenstern Expected Utility Theory was received quite well, it did not escape criticism. The fiercest opponent of the theory was Maurice Allais (1911-2010), a French economist and the Nobel Memorial Prize winner in Economics for contributions to the theory of markets and efficient utilisation of resources. He devoted significant part of his research to decision theory. His critique of so called American School and the Expected Utility Theory had a very important impact on further development of the notion of utility. In 1953 he discovered a systematic violation of Expected Utility Theory, to be precise, the violation of independence axiom. The example he created is known as Allais Paradox. Now I am about to introduce the origin of the Allais Paradox and its formal description.

One of the most common references concerning the Allais Paradox apart from the book "Expected Utility Hypotheses and The Allais Paradox - Contemporary Discussions of Decisions under Uncertainty with Allais' Rejoinder" (Machina, 1995) is his article published in *Econometrica* in October 1953. The article was an offspring of a lively debate on the subject at colloquium held in Paris in May 1952. The article titled *"Le Comportement de l'Homme Rationnel devant le Risque: Critique des Postulats et Axiomes de l'Ecole Americaine"*[19] (Allais, 1953) was preceded by a very interesting editor's note. The editor suggested that at the time of publication of the article, its content was of an extremely subtle sort and it seemed very difficult to reach a general agreement on the points made by Allais. Furthermore, it was suggested by the editor that despite most convenient circumstances at the aforementioned colloquium in Paris, some misunderstandings among the participants were inevitable and could not be clarified to the satisfactory extent. Hence, the article was published on the sole author's responsibility. The note ends with a remark that even though the points made in the article were very fragile, the editor believed that the work would be of a valuable nature and would prevent the isolation of ideas regarding the subject within a very small scientific community.

Maurice Allais starts his article with presenting four factors which in his opinion should be taken into account by every theory regarding risk in order to be realistic.

1. Monetary and psychological values should be clearly distinguished.

2. The problem of the distortion of objective probabilities and the appearance of subjective probabilities.

[19]Own translation: "The behaviour of rational human beings in the face of risk. The critique of postulates and axioms of the American school."

3. The aspect of mathematical expectation of psychological values.

4. The general properties of the probability distribution of psychological values i.a. variance.

The last factor (4) is probably the most important when considering theory of risk. Among the secondary factors the author suggests taking into account the expenses connected with each gamble (choice), the pleasure derived from the gamble itself and the magnitude of the *minimum sensible*[20] principle.

In the following part of the article Maurice Allais questions the axiomatic foundation of, as he refers to it, American school and "the principle of Bernoulli". He claims that everyone knows that in real world people do not abide the rules of rationality developed by Americans. The author admits, however, that the views on how a rational individual should behave are diverse. Nevertheless, he rises a strong objection to defining rationality as adherence to the particular system of axioms which is the basis of, as he refers to it, a Bernoulli type formulation of American school.

According to Maurice Allais, rationality in order to be interesting from the scientific point of view must be defined in one of the two following ways. The former approach suggests that rationality could be defined as an abstract entity by referring to the general criterion of internal consistency which is well defined in the social sciences. This criterion implies the coherence of desired ends and the use of appropriate means to achieve them. The latter, however, suggests to define rationality experimentally by observing the actions of people who are a priori regarded as acting in a rational manner (defined subjectively by the researcher). Personally, I am very sceptical when it comes to the the latter method of defining rationality because the a priori assumption contradicts in a sense the process itself.

The aforementioned principle of internal consistency implies that the objective probabilities should be used whenever they exist. Furthermore, it implies the axiom of absolute preference which stands: when given two choices, one is certainly preferable if, for all possible outcomes, it yields a greater gain. Maurice Allais concludes that these two prerequisites (i.e. principle of internal consistency and the axiom of absolute preference) are less restrictive than the assumptions of American school. Hence, there exist (at least according to the

[20]minimum sensible - a term introduced by Anglo-Irish philosopher George Berkeley (1685 – 1753) which refers to capacity of human sense-impressions. The main assumption of Berkeley was that all objects of an immediate perceptions are sense-impressions. However, the capacity of human senses is finite. Hence, they are not infinitely divisible and must be composed of a finite number of *minimum sensibilia*. In other words, there must exist a minimum tangible or a minimum visible size, beyond which sense cannot perceive.

definition above) rational types of behaviour which do not obey axioms of American school.

Maurice Allais performed numerous experiments on individuals who were considered rational by the public opinion. Results of these experiments showed that people do not necessarily obey every axiom developed by the American school. He indicated four classes of facts which in his opinion were of the most importance regarding this matter. Two of them were connected with gambles of very small and very large sums of money. The author suggested that it is important how very cautious people behave while taking part in gambles of small sums and, on the other hand, what is the behaviour of entrepreneurs when there is a possibility of encountering a great loss. One of the final remarks found in the article considers the use of law of large numbers as justification for the formulation of American school. Maurice Allais stated that for him "it is a pure illusion".

3.3 Indifference curves theory and the formulation of Allais Paradox

Having presented Allais's attitude towards Expected Utility Theory and the origin of Allais Paradox, let us present how the paradox was formulated. However, prior to that, we are about to explain the concept mentioned earlier in the previous chapter i.e. notion of indifference curves. This is necessary to show why the problem formulated by Maurice Allais contradicts the axioms developed by John von Neumann and Oskar Morgenstern.

Let us for a moment focus on the property of linearity in the probabilities which is strictly connected to the independence axiom (it will be discussed in the following paragraphs). Our aim is to graphically illustrate this property. Let $\sum p_i U(x_i)$ denote the preference function, where x_i indicates the payoff, p_i indicates the probability of achieving particular payoff and finally $U(\cdot)$ indicates the utility function. Let us consider a set of all lotteries over the fixed outcome levels such that $x_1 < x_2 < x_3$. Let us notice that these outcomes can be represented by the set of all probability triples of the form $P = (p_1, p_2, p_3)$ where p_i denotes the probability of the payoff x_i and of course $\sum p_i = 1$. Hence, $p_2 = 1 - p_1 - p_3$. Now we can represent these three lotteries by the points in the unit triangle on the plane (p_1, p_3). Such triangle is called Marschak-Machina Triangle and is shown in the figure 3 on page 51 (Machina, 1987).

The Marschak-Machina Triangle in this example works in the following way:

- **upward movements** in the triangle increase p_3 at the expense of p_2 (in other words, the probability is shifted from the outcome x_2 up to x_3)

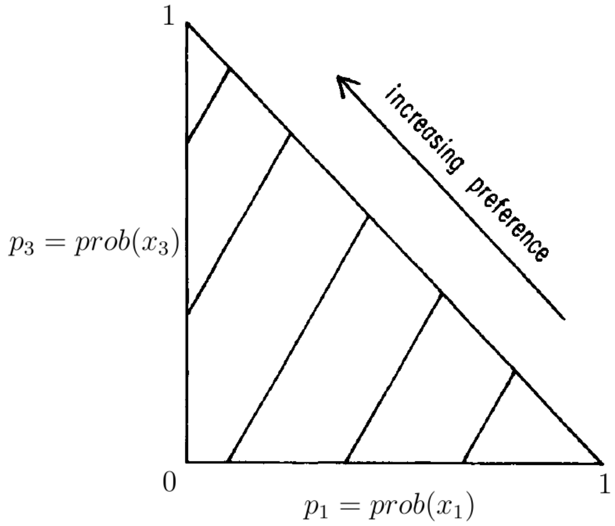

Figure 3: Marschak-Machina Triangle. Source: (Machina, 1987)

- **leftward movements** reduce p_1 to the benefit of p_2 (the probability is shifted from x_1 up to x_2)
- **upward and leftward movements** (or more generally - **northwest movements**) lead to dominating lotteries and would be preferred accordingly

The indifference curves of the individual on the plane (p_1, p_3) are given by the solutions to the following linear equation:

$$\bar{u} = \sum_{i=1}^{3} p_i U(x_i) = p_1 U(x_1) + (1 - p_1 - p_3) U(x_2) + p_3 U(x_3) = constant \quad (3.1)$$

Hence, the indifference curves will consist of parallel straight lines of slope

$$\frac{U(x_2) - U(x_1)}{U(x_3) - U(x_2)}$$

where more preferred indifference curves lie to the northwest. Marschak-Machina Triangle can be used to illustrate different attitudes toward risks i.e. risk-aversion and risk-seeking. The dashed lines in the figure 4 and figure 5 on page 53 denote so called iso-expected value lines (these are not indifference curves!) which represent the solutions to the following equation:

$$\bar{x} = \sum_{i=1}^{3} p_i x_i = p_1 x_1 + (1 - p_1 - p_3) x_2 + p_3 x_3 = constant \quad (3.2)$$

It is easy to notice that northeast movements along these lines do not change the expected value, however, they increase the probabilities of the outcomes x_1 and x_3 at the expense of the outcome x_2.[21] The risk interpretation in the Marschak-Machina triangles should be based on the assumption of fixed outcome levels (i.e. $x_1 < x_2 < x_3$). Let us examine two cases:

- **RISK AVERSION** - utility function $U(\cdot)$ is **concave** and indifference curves are steeper than the iso-expected value lines and increases in risk will lead to lower indifference curves. See figure 4 for graphical representation.

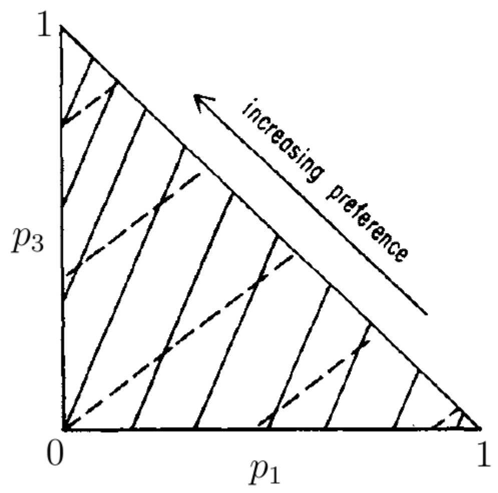

Figure 4: RISK AVERSION - relatively steep indifference curves. Source: (Machina, 1987)

- **RISK SEEKING** - utility function $U(\cdot)$ is **convex** and indifference curves are flatter than the iso-expected value lines and increases in risk will lead to higher indifference curves. See figure 5 on page 53 for graphical representation.

It can be explained by noticing that:

- the slope of the indifference curves is given by

$$\frac{U(x_2) - U(x_1)}{U(x_3) - U(x_2)}$$

- the slope of the iso-expected value lines is given by

$$\frac{x_2 - x_1}{x_3 - x_2}$$

[21] They are examples of *mean preserving spreads* or *pure increases in risk*.

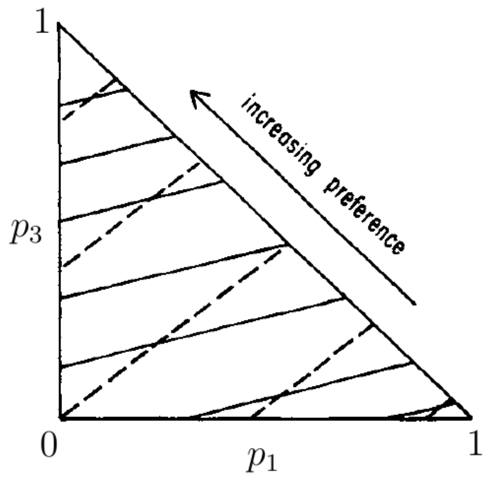

Figure 5: RISK SEEKING - relatively flat indifference curves. Source: (Machina, 1987)

- concavity of $U(\cdot)$ implies

$$\frac{U(x_2) - U(x_1)}{x_2 - x_1} > \frac{U(x_3) - U(x_2)}{x_3 - x_2}$$

whenever $x_1 < x_2 < x_3$.

To sum up, while comparing two different utility functions, the one which is more risk averse will possess the steeper indifference curves.

Let us remind the definition of independence axiom (5.) in the form presented in the definition 6 on page 41.

$$\lambda_1 \succ \lambda_2 \Rightarrow q\lambda_1 + (1-q)\lambda_3 \succ q\lambda_2 + (1-q)\lambda_3$$

If the lottery λ_1 is preferred to the lottery λ_2, than the mixture $q\lambda_1 + (1-q)\lambda_3$ will be preferred to the mixture $q\lambda_2 + (1-q)\lambda_3$ for all $q > 0$ and λ_3.

This axiom is in fact equivalent to linearity in the probabilities. In order to understand it better let us present the following interpretation.

Let us imagine that an individual is being offered a toss of a special coin. The probability of this coin landing tails is equal to $(1-q)$ (then the probability of landing heads is obviously equal to q). Whenever it lands tails, the individual obtains a lottery λ_3. Prior to the toss, the individual is asked about his preference of two lotteries λ_1 and λ_2 in case of the coin landing heads. Let us consider two possible outcomes of such situation. If the coin lands tails, the choice of the individual made before the toss does not matter since he is given the lottery λ_3. However, if it lands heads, the individual is back "in a sense" to a choice between lotteries λ_1 and λ_2. Now, if given such choice after it landed heads, the only 'rational' decision would be to make the same choice as it was made prior to the toss.

Although the argument above played a major role in adoption of expected utility concept as a descriptive theory of choice under uncertainty in economics, it also created a tension between economists. Now let us present the Allais Paradox itself which is one of the earliest and at the same time best known examples of systematic violation of independence axiom.

In 1953 Maurice Allais performed an experiment in which he asked respondents to make two choices (Choice 1 and Choice 2). The first one was to choose a preferable option from a certain option A_1 or probable option B_1. The second choice was to choose from two probable options A_2 and B_2. The experiment is summarised by the table below showing each choice, the probabilities and potential payoffs (in millions of $).
Let us kindly remark that if the reader is not familiar with the paradox, it is advisable to make the choice from the presented options before proceeding with the reading further.

Choice 1				Choice 2			
Option A_1		Option B_1		Option A_2		Option B_2	
Payoff	Probability	Payoff	Probability	Payoff	Probability	Payoff	Probability
1	1,00	1	0,89	0	0,89	0	0,90
		0	0,01	1	0,11		
		5	0,10			5	0,10
	$\sum = 1$		$\sum = 1$		$\sum = 1$		$\sum = 1$
$\mathbb{E}(A_1) = 1$		$\mathbb{E}(B_1) = 1,39$		$\mathbb{E}(A_2) = 0,11$		$\mathbb{E}(B_2) = 0,50$	

Table 3: Original options in Allais Paradox (Payoffs in millions of $). Source: *Own compilation.*

Maurice Allais found out that majority of respondents chose option A_1 from Choice 1 (Option A_1 or B_1) and option B_2 from Choice 2 (Option A_2 or B_2). This result (i.e. choosing A_1 and B_2) contradicts the independence axiom of Expected Utility Theory, since according to it, choosing option A_1 from Choice 1 should imply the choice of option A_2 from Choice 2 and analogically choice of option B_1 from Choice 1 should imply the choice of option B_2 from Choice 2. In the following paragraphs two simple explanations why the independence axiom is violated will be shown, however, prior to that let us modify the Table 3 so as it is possible to see the analogy to the example of special coin given above.
The inconsistency in the above example stems from the fact that according to the independence axiom of Expected Utility Theory, any additional outcomes

Choice 1				Choice 2			
Option A_1		Option B_1		Option A_2		Option B_2	
Payoff	Probability	Payoff	Probability	Payoff	Probability	Payoff	Probability
1	0,89	1	0,89	0	0,89	0	0,89
1	0,11	0	0,01	1	0,11	0	0,01
		5	0,10			5	0,10
	$\sum = 1$		$\sum = 1$		$\sum = 1$		$\sum = 1$
$\mathbb{E}(A_1) = 1$		$\mathbb{E}(B_1) = 1,39$		$\mathbb{E}(A_2) = 0,11$		$\mathbb{E}(B_2) = 0,50$	

Table 4: Modified options in Allais Paradox (Payoffs in millions of $). Source: *Own compilation.*

(which are identical) added to each possible choice should not affect in any way the ultimate choice in a given lottery (gamble). These additional outcomes should "cancel out" in a sense speaking informally. It can be easily noticed by looking at the modified version of the original choices presented in the Table 4. When the fourth row from the Table 4 is examined, it can be concluded that in options A_1 and B_1 from Choice 1 exists the same constituent i.e. payoff=1 with probability=0,89. Similar observation can be made about the Choice 2. Options A_2 and B_2 from this choice also include identical constituent which in this case is payoff=0 with probability=0,89. When it is assumed that these constituents "cancel out", the individual is left with exactly the same options in both choices (Choice 1 and Choice 2) which is illustrated by the Table 5.

Choice 1				Choice 2			
Option A_1		Option B_1		Option A_2		Option B_2	
Payoff	Probability	Payoff	Probability	Payoff	Probability	Payoff	Probability
1	0,11	0	0,01	1	0,11	0	0,01
		5	0,10			5	0,10

Table 5: "Cancelled out" options in Allais Paradox (Payoffs in millions of $). Source: *Own compilation.*

Hence, the individual should not have any inclination to change the preferred option in Choice 2. In other words, to stay consistent, the choice of option A_1 should imply the choice of option A_2 and analogically the choice of option B_1 should imply the choice of option B_2. The experiment showed that this is not preserved, hence, the independence axiom is violated.

Now let us proceed with a more formal explanations of why independence axiom is violated. The first explanation will utilise the tables above and compare some inequalities with utility functions. The majority of respondents chose option A_1 in Choice 1 and option B_2 in Choice 2. The expected utilities of options A_1 and

B_1 can be expressed by the following inequalities:

$$\underbrace{U(1)}_{\text{Option } A_1} \geqslant \underbrace{0,01 \cdot U(0) + 0,89 \cdot U(1) + 0,10 \cdot U(5)}_{\text{Option } B_1} \tag{3.3}$$

The above equation (3.3) is based on the table 3. However, we will simplify it according to the table 5:

$$\underbrace{0,11 \cdot U(1)}_{\text{Option } A_1} \geqslant \underbrace{0,01 \cdot U(0) + 0,10 \cdot U(5)}_{\text{Option } B_1} \tag{3.4}$$

Analogically, the expected utilities of options A_2 and B_2 can be expressed by the following inequalities:

$$\underbrace{0,89 \cdot U(0) + 0,11 \cdot U(1)}_{\text{Option } A_2} \leqslant \underbrace{0,90 \cdot U(0) + 0,10 \cdot U(5)}_{\text{Option } B_2} \tag{3.5}$$

The simplification according to the table 5:

$$\underbrace{0,11 \cdot U(1)}_{\text{Option } A_2} \leqslant \underbrace{0,01 \cdot U(0) + 0,10 \cdot U(5)}_{\text{Option } B_2} \tag{3.6}$$

The inconsistency in the above equations can be noticed by looking at the inequalities 3.4 and 3.6. They both are the same and the only difference is that the inequality sign is reversed. Hence, it shows the violation of independence axiom of the Expected Utility Theory. This explanation requires additional remark that all of the above inequalities are valid regardless of the type of utility function (risk-averse, risk-neutral, risk-seeking). This is due to the fact that the type of risk propensity is determined by the second derivative of the utility function (which can be negative, zero or positive for risk-averse, risk-neutral and risk-seeking individuals respectively) which is not present in any of the above inequalities. Hence, it is concluded that the the axiom is violated regardless of the type of utility function (which by definition must be unique up to increasing monotonous transformation).

The second formal explanation is connected with previously explained technique of indifference curves. Firstly, let us notice that in the Allais Paradox there are only three possible payoffs (considering all possible options A_1, B_1, A_2 and B_2 from both choices, the only possible payoffs are: 0m \$, 1m \$ and 5m \$) with different probabilities. Therefore, let the set of fixed outcome levels be defined as follows $\{x_1, x_2, x_3\} = \{\$0, \$1\ 000\ 000, \$5\ 000\ 000\}$. Then, the four options A_1, B_1, A_2 and B_2 form a parallelogram in the (p_1, p_2) triangle as shown in the figure 6.

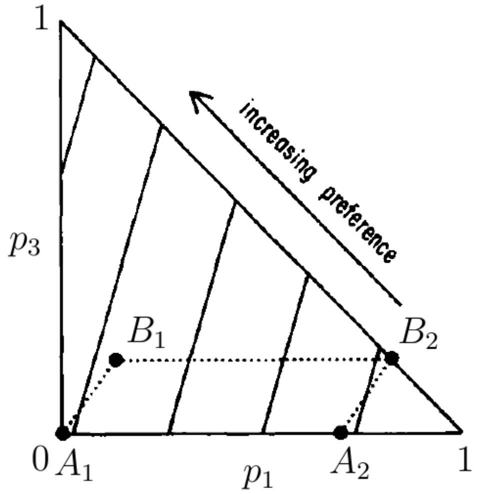

Figure 6: Expected utility indifference curves. Source: (Machina, 1987)

On the ground of The Expected Utility Theory, the preference of Option A_1 in the Choice 1 indicates that the indifference curves should be relatively steep and, therefore, it implies the preference of option A_2 in the Choice 2. Analogically, if Option B_1 is selected by the individual, than the indifference curves are relatively flat and imply the selection of option B_2 in the second choice. The already discussed result of Allais experiment, which shows the systematic violation of independence axiom, implies that indifference curves are not parallel but rather fan out in this case. This fact is presented graphically in the figure 7.

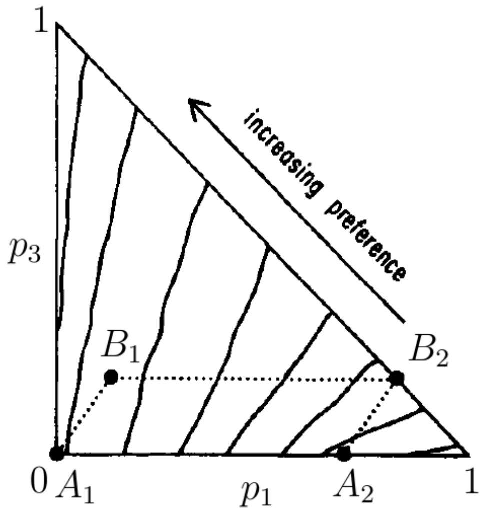

Figure 7: "Fanning out" indifference curves. Source: (Machina, 1987)

3.4 Various psychological causes of Expected Utility Hypothesis violation

Allais Paradox is not the only systematic violation of Expected Utility Hypothesis which is based on risk aversion. Since the publication of the paradox, many different violations have been noted on various grounds including psychology (Mongin, 1997) (Rios, et al. 1997). Only a few most important concepts from the literature will be presented in this thesis together with a brief description i.a. The Certainty Effect, The Big Amount Effect, The Common Consequence Effect, The Common Ratio or Isolation Effect, The Reverse Common Ratio Effect, The Response Mode Effect and The Framing Effect.

3.4.1 The Certainty Effect

The Certainty Effect together with The Common Consequence Effect and The Common Ratio Effect are probably the most recognisable violations of the Expected Utility Hypothesis apart from Allais Paradox. The Certainty Effect was developed by Daniel Kahneman (1934-) - psychologist and behavioural economists, together with Amos Tversky (1937-1996) - a psychologist, in 1979. Khaneman and Tversky mainly known for the development of Prospect Theory (which lies beyond the scope of this thesis) conducted a comprehensive experimental research close to the concept of Allais Paradox. Not only did their research verify Allais Paradox but it also led them to a new conclusion. The most characteristic feature of they research was the fact that some options in presented questions contained almost certain instead of fully certain payoffs. The authors concluded from the research that there exists a tendency which causes individuals to underweight probable payoffs when it is possible to compare them with certain or at least almost certain payoffs. They called such tendency - the certainty effect. It reinforces risk aversion when available options contain only positive payoffs and, on the other hand, reinforces risk seeking when available options contain non-positive (zero) payoffs. Let us present the example from Khaneman and Tversky research in the table 6 on page 59 which illustrates The Certainty Effect (payoffs in $).
The rules of the problem presented in the table above were exactly the same as in Allais Paradox. The result was also similar as majority of respondents selected options A_1 and B_2 as in Allais Paradox. The choice A_1 together with B_2 violated Expected Utility Theory due to the certainty effect as payoff in option A_1 was certain.

The certainty effect is not limited to cases in which the payoff in available options is absolutely certain but it can also be extended into cases in which the

Choice 1				Choice 2			
Option A_1		Option B_1		Option A_2		Option B_2	
Payoff	Probability	Payoff	Probability	Payoff	Probability	Payoff	Probability
3000	1,00	0	0,20	0	0,75	0	0,80
		4000	0,80	3000	0,25	4000	0,20
	$\sum = 1$		$\sum = 1$		$\sum = 1$		$\sum = 1$

Table 6: The Certainty Effect (1) based on Khaneman and Tversky problems 3 and 4 (Kahneman and Tversky, 1979). Source: *Own compilation.*

payoff of available options is almost certain. The next table (table 7 below) presents another problem from Khaneman and Tversky research which illustrates the almost certainty effect.

Choice 1				Choice 2			
Option A_1		Option B_1		Option A_2		Option B_2	
Payoff	Probability	Payoff	Probability	Payoff	Probability	Payoff	Probability
0	0,10	0	0,55	0	0,998	0	0,999
3000	0,90	6000	0,45	3000	0,002	6000	0,001
	$\sum = 1$		$\sum = 1$		$\sum = 1$		$\sum = 1$

Table 7: The Almost Certainty Effect (2) based on Khaneman and Tversky problems 7 and 8 (Kahneman and Tversky, 1979). Source: *Own compilation.*

Similarly to the previous example, the respondents again selected the pair of options A_1 and B_2. The Expected Utility Hypothesis was violated due to the almost certainty effect. Let us look closer into the presented example. It is worth noticing that while in the Choice 1 in Option A_1 the only positive payoff was almost certain with probability 0,90 and in Option B_1 the only positive payoff had probability 0,45, the only positive payoffs in Choice 2 (both in option A_2 and B_2) were merely probable with probabilities 0,002 and 0,001 respectively. What might not be visible at first sight is the fact that probabilities of positive payoffs in Choice 2 were simply scaled down (in proportion 1/450) probabilities of positive payoffs in Choice 1, hence, it should not affect the choice of respondents according to Expected Utility Hypothesis.

3.4.2 The Big Amount Effect

The Big Amount Effect is probably the most commonly observed effect in everyday life. It is mainly connected with national lotteries, gambling and, on the other hand, insurance. The effect was firstly described by Milton Friedman (1912 – 2006) - American economist, and Leonard J. Savage (1917 – 1971) - American mathematician. However, it was later elaborated by aforementioned Daniel Kahneman and Amos Tversky (Weber, 2008). Kahneman and Tversky stated that people tend to overestimate the expected value of lotteries which offer small probability of winning large amounts of money. A perfect example of such behaviour is buying a lottery ticket for national lottery[22] where the odds of winning the highest prize are 1 to 13 983 816. This effect reinforces risk-seeking attitude and can be an explanation of why individuals willingly pay a fee (or equivalently - buy lottery tickets) which price is higher than the actual expected value of the lottery. Friedman and Savage said about such individual that he prefers large chance of losing a small amount (the lottery fee/ticket) and small chance of winning a big amount (the prize of a lottery) to avoidance of both of these risks (to keep the price of a lottery fee/ticket). In other words, he willingly chooses uncertainty to certainty in hope of winning a big amount (prize of a lottery), even though he is aware of the fact that on average the expected value of the lottery is lower than the price of a fee/ticket to the lottery itself. The same can be said about individuals who buy fire insurance on house. In such situation, an individual prefers paying a fixed sum (the insurance premium) to being exposed to a small chance of a big loss (the value of the house) and a large chance of no loss. In other words, the individual in this case prefers certainty to uncertainty in order to avoid potential big loss, even though, he is aware of the fact that the insurance premium is, on average, greater than the expected value of potential loss (burning of the house).

3.4.3 The Common Consequence Effect

The Common Consequence Effect is a general empirical pattern, a specific example of which is Allais Paradox. Let us go back to the tables describing the paradox in order to explain what the common consequence effect really is. The table 9 (a copy of table 4 from page 55) which was the modification of table 8 (a copy of table 3 from page 54) showed us the existence of identical implicit payoffs hidden in the options of each choice (payoff=1 with probability=0,89 for Choice 1 and payoff=0 with probability=0,89 for Choice 2). These payoffs are called implicit since in the original table 8 they are not showed explicitly - they are visible only after the slight modification in table 9. Now, let us look closer at inequalities 3.7 and 3.8 (both inequalities being a copy of inequalities 3.4 and 3.6 from page 56) below.

$$\underbrace{0,11 \cdot U(1)}_{\text{Option } A_1} \geqslant \underbrace{0,01 \cdot U(0) + 0,10 \cdot U(5)}_{\text{Option } B_1} \tag{3.7}$$

[22]The example considered here is based on Polish national lottery "Lotto"

$$\underbrace{0,11 \cdot U(1)}_{\text{Option } A_2} \leqslant \underbrace{0,01 \cdot U(0) + 0,10 \cdot U(5)}_{\text{Option } B_2} \tag{3.8}$$

They are simplified versions of inequalities 3.9 and 3.10 (both inequalities being a copy of inequalities 3.3 and 3.5 from page 56) respectively.

$$\underbrace{U(1)}_{\text{Option } A_1} \geqslant \underbrace{0,01 \cdot U(0) + 0,89 \cdot U(1) + 0,10 \cdot U(5)}_{\text{Option } B_1} \tag{3.9}$$

$$\underbrace{0,89 \cdot U(0) + 0,11 \cdot U(1)}_{\text{Option } A_2} \leqslant \underbrace{0,90 \cdot U(0) + 0,10 \cdot U(5)}_{\text{Option } B_2} \tag{3.10}$$

Inequality 3.7 was achieved by subtracting $0,89 \cdot U(1)$ from inequality 3.9, and similarly, inequality 3.8 was achieved by subtracting $0,89 \cdot U(0)$ from inequality 3.10. The subtracted elements i.e. $0,89 \cdot U(1)$ and $0,89 \cdot U(0)$ are called common consequences. They can be interpreted as follows. The common consequence with higher expected value (in our case i.e. $0,89 \cdot U(1)$) makes respondents risk-averse in Choice 1 while the common consequence with lower expected value (i.e. $0,89 \cdot U(0)$) in Choice 2 makes them risk-seekers.

Choice 1				Choice 2			
Option A_1		Option B_1		Option A_2		Option B_2	
Payoff	Probability	Payoff	Probability	Payoff	Probability	Payoff	Probability
1	1,00	1	0,89	0	0,89	0	0,90
		0	0,01	1	0,11		
		5	0,10			5	0,10
	$\sum = 1$		$\sum = 1$		$\sum = 1$		$\sum = 1$
$\mathbb{E}(A_1) = 1$		$\mathbb{E}(B_1) = 1,39$		$\mathbb{E}(A_2) = 0,11$		$\mathbb{E}(B_2) = 0,50$	

Table 8: Original options in Allais Paradox (Payoffs in millions of $). Source: *Own compilation.*

Choice 1				Choice 2			
Option A_1		Option B_1		Option A_2		Option B_2	
Payoff	Probability	Payoff	Probability	Payoff	Probability	Payoff	Probability
1	0,89	1	0,89	0	0,89	0	0,89
1	0,11	0	0,01	1	0,11	0	0,01
		5	0,10			5	0,10
	$\sum = 1$		$\sum = 1$		$\sum = 1$		$\sum = 1$
$\mathbb{E}(A_1) = 1$		$\mathbb{E}(B_1) = 1,39$		$\mathbb{E}(A_2) = 0,11$		$\mathbb{E}(B_2) = 0,50$	

Table 9: Modified options in Allais Paradox (Payoffs in millions of $). Source: *Own compilation.*

3.4.4 The Common Ratio or Isolation Effect

The Common Ratio or Isolation Effect is strictly connected with The Certainty Effect. The empirical evidence of the common ratio effect was proven in the research of Kahneman and Tversky mentioned earlier while discussing the certainty effect. It has been noticed by the researchers that majority of individuals tend to choose certain or almost certain option in the original version of presented problem and risky option in the new problem which is in fact a slightly modified version of the original problem with scaled down probabilities. Numerical examples of the common ratio effect are very similar to the research carried out by Kahnemann and Tversky, hence, only the theoretical background concerning the effect will be presented.

Let a problem be constructed as Allais Paradox i.e. the problem consists of two choices - Choice 1 between two options: A_1 and B_1; and Choice 2 between two options: A_2 and B_2. Let an option A_1 from Choice 1 consist of certain outcome X or let A_1 consist of two outcomes: almost certain outcome X and 0. Secondly, let an option B_1 consist of two probable (neither certain nor almost certain) outcomes: Y and 0. Finally, we require that $Y > X > 0$ and $1 > p_X > \frac{1}{2} > p_Y > 0$ where p_X and p_Y denote probabilities of outcomes X and Y respectively. The second choice (Choice 2) and options associated with it (A_2 and B_2) are created by scaling down the probabilities of X and Y from the Choice 1 by the same proportion k such that the ratio $\frac{p_X \cdot X}{p_Y \cdot Y}$ remains the same (common) in both choices (Choice 1 and Choice 2). See the table 10 below. Then, according to Expected Utility Theory, the respondents should not change their choices between Choice 1 and Choice 2.

Choice 1				Choice 2			
Option A_1		Option B_1		Option A_2		Option B_2	
Payoff	Probability	Payoff	Probability	Payoff	Probability	Payoff	Probability
X	p_X	Y	p_Y	X	$p'_X = k \cdot p_X$	Y	$p'_Y = k \cdot p_Y$
0	$1 - p_X$	0	$1 - p_Y$	0	$1 - p'_X$	0	$1 - p'_Y$
	$\sum = 1$		$\sum = 1$		$\sum = 1$		$\sum = 1$

Table 10: The Common Ratio or Isolation Effect. Source: *Own compilation.*

3.4.5 The Reverse Common Ratio Effect

Let the problem be constructed in a similar way to the problem described above in the description of The Common Ratio Effect. Let an option A_1 from Choice 1 consist of certain outcome X and let an option B_1 from the same Choice 1 consist of two probable outcomes: Y and 0. We require that $Y > X > 0$. Then, the Choice 2 is constructed analogically to the previous case by scaling down the probabilities of outcomes X and Y by the same proportion such that the ratio $\frac{p_X \cdot X}{p_Y \cdot Y}$ remains the same (common) in both choices (Choice 1 and Choice 2). However, we require that probabilities after transformation (scaled-down) p'_x and p'_Y meet the following requirement: $\frac{1}{2} > p'_X > p'_Y > 0$. According to Professor Pavlo Blavatskyy (Blavatskyy, 2010), when the problem is constructed in such a way, respondents show exactly the opposite behaviour than this described in the common ratio effect above. Hence, the name Reverse Common Ratio Effect. In other words, the majority of respondents choose risky option in original version of the problem (Choice 1) and safer option in modified version (Choice 2) with scaled down probabilities.

Blavatskyy in his research also found a correlation between the amount of certain option and the expected value of probable option. He concluded that respondents are pre-programmed in a sense since their decisions depend upon this correlation. When the amount of certain option is slightly lower than the expected value of probable option, then the majority of respondents chooses certain option in Choice 1. On the other hand, if the amount of certain option is far lower than the expected value of risky option, the majority of respondents chooses risky option.

3.4.6 The Response Mode Effect

Having presented The Certainty Effect and The Big Amount Effect, let us present The Response Mode Effect observed by Lichtenstein (1971) and Slovic (1973). They performed many hypothetical experiments and empirical research in actual casinos. What they found out was another systematic violation of Expected Utility Theory which they called The Response Mode Effect. The scientists observed that majority of respondents pay more attention to probabilities of particular payoffs or to the certainty effect when they are asked to rank or choose from two competing gambling options. However, when they are asked to bid or ask money for the same options they tend to pay more attention to the amount of these payoffs or to the big amount effect. As a result, in the first option they choose the most probable payoff even though the overall expected value of the chosen option might be lower than of the competing one. When the latter case is considered, they bid or ask a bigger amount of money for the option with the highest payoff despite its expected value which might be lower than of the other option. The authors explain this behaviour patter by pointing out that human information processing system focuses more on probability while ranking or choosing from competing probable options and more on payoffs when assessment of monetary value is considered.

The reversal of preference was confirmed in 1990 by Slovic, Kahneman and Tversky. They showed a simple example in which respondents were asked to choose from two options - option A called "P-bet" and option B called "$-bet". In the former option the respondents had a chance with probability $28/36 \approx 0,78$ to win $10 or nothing ($0) otherwise (with remaining probability $1-(28/36) \approx 0,22$). The latter option involved a chance with probability $3/36 \approx 0,08$ of winning $100 or nothing ($0) otherwise (with remaining probability $1-(3/36) \approx 0,92$). Let us point out that expected values of option A (P-bet) and option B ($-bet) were 7,78 and 8,33 respectively ($\mathbb{E}(A) = 7,78 < \mathbb{E}(B) = 8,33$). Despite the fact that expected value of the former option was lower than of the latter option, the majority of respondents chose option A. However, when asked about the lowest selling price of the same options, the majority of respondents stated a higher price for option A even though variance of option B was many times higher that this of option A ($Var(B) = 763,89 \gg Var(A) = 17,8$).

3.4.7 The Framing Effect

Another reason for violating the Expected Utility Theory is The Framing Effect researched by many authors i.a. Slovic (1969), Payne and Braunstein (1971), Khaneman and Tversky (1979), Hershy and Schoemaker (1980). Let us present this effect on the example of Khaneman an Tversky from 1979. The respondents were asked to imagine that they were given $1000 regardless of they current wealth. Next, they were asked to choose between the certain (sure) payoff of $500 and a chance (gamble) of getting $1000 with a probability 0,5 or nothing ($0) with remaining probability 1-0,5=0,5. Majority of the respondents chose the sure payoff. Secondly, the respondents were asked to imagine that they were given $2000 regardless of their current wealth. After that they were asked to make a choice between a certain (sure) loss of $500 or a chance (gamble) of losing $1000 with a probability of 0,5 or losing nothing at all otherwise. This time, majority of the respondents preferred the gamble. When the both problems are compared, its is easy to notice that the second problem is not much different from the first one when financial position of the respondents is concerned, however, the choices they made are different. Khaneman and Tversky concluded that it is due to the framing effect which causes people to be risk-averse in case of positive payoffs and risk-seeking in case of the negative ones. This is a very important psychological notion which affects decision making.

Conclusions

The brief description of the most important psychological causes of Expected Utility Theory violation concludes the thesis. It was mainly focused on presenting the roots of the Expected Utility Hypothesis (St. Petersburg Paradox and Pascal's Wager) and most important and well known notions associated with it (Von Neumann–Morgenstern Expected Utility Theory and Allais Paradox). Several reasons for such a structure of the thesis can be found. First and foremost, the topic of Expected Utility Hypothesis is extremely broad - ranging from St. Petersburg Paradox discovered over three hundred years ago to a fairly new Cumulative Prospect Theory (CPT) developed in 1992 by Amos Tversky and Daniel Kahneman. Hence, the scope of this work must have been limited in order to avoid potential superficial treatment of the subject. It was decided to focus on the early development of the theory and its most recognised critique due to a very important characteristic of the literature concerning the problem. During the research and literature overview performed prior to writing this thesis, it was found that despite the wide range of available English literature on the topic, it is characterised by high complexity and is mainly directed to highly specialised researchers and scientists in this field (e.g. (Schoemaker, 1980), (Hagen and Wenstop, 1984), (Krelle, 1984)). Substantial majority of the articles and books found, assumed that the reader is absolutely familiar with all the concepts mentioned in this thesis. Furthermore, no scientific publication was found concerning the basics of the subject or covering most of the problems included in the thesis in a single work. Moreover, the first publications of the notions mentioned in this work were usually a bit archaic or consisted of complex and meticulous mathematical considerations. Therefore, it was considered a worthy challenge to gather, analyse, synthesise and systematise the selected literature on the subject in such a way to present a comprehensive, thorough and accessible to a less experienced or knowledgeable readers in this field a worth reading material. Such a task required introduction of some elementary mathematical notions, however, it was limited only to the essential ones in order to preserve the economic perspective of the thesis.

Due to the chosen approach of writing this work, some very important, and at the same time interesting notions, were not included. One of them was Subjective Expected Utility model axiomatised by an American mathematician and statistician Leonard Jimmie Savage (1917 - 1971) which relied on two concepts: the utility function (as in Expected Utility Theory) and a personal probability distribution (based on Bayesian probability theory). However, this model was also questioned and challenged by another paradox - Ellsberg

Paradox which violated its postulates. Finally, very important models of Kahneman and Tversky such as Prospect Theory and Cumulative Prospect Theory could not have been described in detail in this thesis.

The Expected Utility Hypothesis topic is still being investigated and developed further by economists, behavioural economists and mathematicians. Furthermore, many doctoral dissertation around the world are focused on developing the idea of expected utility. However, it seems like it is not in the mainstream of the current economic researches and is not considered a very popular subject outside United States of America. For instance, in Poland, only a handful of publications regarding this notion can be found and, what is more, they rarely extend or try to develop it further. One of the reasons for that might be the relatively young age of the theory and constant changes being alternately introduced and challenged by the scientists. However, it is worth noticing that the notion of expected utility yields great potential in applying it to practical economic problems (hence, it is a subject of research of behavioural economics).

The thesis has some unique aspects such as its particular scope and composition. It presents the subject in a comprehensive and logical way. Not many works present the problem by relating to the roots of it. However, this work gives the reader a complete background overview by referring to the earliest notions of 'infinite gain' such as the problem of Pascal's Wager. Subsequently, it considers the early development of Expected Utility Hypothesis with St. Petersburg Paradox. Finally, description of contemporary psychological aspects related to the violation of core assumptions of the theory is presented. They yield a promising future consequences and development of the theory. The thesis presents the problem in a modern setting and gives a comprehensive guidance for readers intending to sort out their knowledge on the subject and prepare for further reading regarding Expected Utility Hypothesis. This work is not the end of my interest in this subject and hopefully I will have a chance to research the problem further in the near future.

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